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A General Umbral Calculus in Infinitely Many Variables*

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1. INTRODUCTION

Thus far, the calculus of finite differences in infinitely many variables has remained largely undeveloped, though interesting problems and potential applications are not lacking in the literature.

In a previous paper [4] the authors introduced a module of functions (factorial functions) which seems to be the appropriate setting for a combinatorial approach to this kind of problem.

Furthermore, these functions provide a ring which, over a field of characteristic zero, is isomorphic to the ring of polynomial functions, but which—by systematic use of binomial coefficients in place of powers—avoids some of the pitfalls in fields of positive characteristic.

The purpose of the present paper is to extend methods and results of [3, 4] to the infinitely many variables case. To this aim, we study sequences of factorial functions satisfying analogs of the well-known binomial and Sheffer recursions and show how an operator calculus resembling the Umbral Calculus can be constructed over them. However, we recognize that an effective analog of closed forms exists only for a class of recursive bases which can be algebraically characterized as those recursive bases whose

* This research was done in strict collaboration. It is therefore impossible to distinguish the conributions of each author, barring Section 9, which is essentially due to the second author.

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graduation is homogeneous with the filtration induced by the difference operators. Binomial coefficients, which we call Pascal functions, and divided powers are examples of such bases. On the other hand, these bases have an analogous characterization by replacing the set of difference operators with some other delta set of shift–invariant operators, like, e.g., partial derivatives. This fact leads us to single out the notion of coherence between a delta set of operators and a recursive basis, which generalizes the situations mentioned above; the algebraic notion of coherence is shown to be the crucial condition to build a substantial calculus, indeed extending the Umbral Calculus in all its computational devices.

Our tools are essentially Laurent series in infinitely many variables and the theory of recursive matrices. In Sections 2 and 3 we give the main definitions and develop the basic properties of Laurent series, their continuous endomorphisms (L-sets), and recursive matrices, respectively. This theory has been previously developed in the single-variable case [3] and it is here extended to this more general setting. The main result of Section 3 is a simple version of the Lagrange–Good formula for infinite sets of Laurent series in infinitely many variables with coefficients ranging over any commutative integral domain.

The module $\mathscr{F}$ of factorial functions is defined in Section 4. Then we associate to any recursive sequence in $\mathscr{F}$ a set of power series, the indicator, and study the relations between algebraic properties of the sequences and combinatorial properties of their indicators. The operator calculus is introduced in Section 5. We examine in details recursive operators, namely, linear operators which map any recursive basis into a recursive sequence of factorial functions, and the subsystem of shift-invariant operators, which turns out to be a commutative topological algebra isomorphic to the algebra of formal power series.

The idea of coherence appears in Section 6. We reconsider the results of Section 5 in the light of this further notion: in particular, we show that the coefficient matrix of a coherent recursive basis with respect to another is the Wiener–Hopf truncation of an invertible recursive matrix, and vice-versa. Finally, in Section 7 it is shown that recursive bases admitting an analog of the transfer formula of the classical Umbral Calculus are just coherent recursive bases. From this combitorial characterization, it is an easy step to generalize the recurrence formula and the transfer formula as simple consequences of our Lagrange–Good type theorems of Section 3.

The last two sections are devoted to some applications. First of all, we describe how the present theory recovers the multivariate Umbral Calculus as developed in recent years by many different authors (see, e.g., [10, 17–19, 22, 23, 26]).

Finally we deal with a classical example: we choose from the literature [1] the theory of multivariate Hermite polynomials and succeed in proving general versions of sundry theorems, such as recurrence relations formulas of Rodrigues type and the Burchnall–Feldheim–Watson identity.

2. The Algebra of Laurent Series

Let S be a set of any cardinality: a map $\mathbf{d}: S \to \mathbb{Z}$ will be called a *degree* whenever its support $\operatorname{supp}(\mathbf{d})$ is finite. The zero degree will be denoted by $\mathbf{0}$. For every $s \in S$, the degree $\mathbf{e}_s$ is defined by setting

$$\mathbf{e}_s(t) := \delta_{st}, t \in S$$

where δ_{st} is the Krönecker symbol.

A degree $\mathbf{d}$ will be said to be a *positive degree* if $\mathbf{d} \not\equiv \mathbf{0}$ and $\mathbf{d}(s) \geqslant 0$, for every $s \in S$.

We will denote by $\mathbf{D}$ the additive group of all degrees, while $\mathbf{D}^+$ will be the monoid of all positive degrees, together with the zero degree. Obviously, $\mathbf{D}$ and $\mathbf{D}^+$ are the free abelian group and the free abelian monoid generated by S, respectively.

$\mathbf{D}$ is naturally structured as an ordered abelian group by setting $\mathbf{d} \leqslant \mathbf{e}$ whenever $\mathbf{e} - \mathbf{d} \in \mathbf{D}^+$. If $\mathbf{d}, \mathbf{e} \in \mathbf{D}^+$ and $\mathbf{d} \leqslant \mathbf{e}$, then $\operatorname{supp}(\mathbf{d}) \subseteq \operatorname{supp}(\mathbf{e})$. With the partial order just defined, $\mathbf{D}$ becomes a locally finite distributive lattice.

The *weight* of a degree $\mathbf{d}$ is the integer

$$w(\mathbf{d}) := \sum_{s \in S} \mathbf{d}(s).$$

The map $w: \mathbf{D} \to \mathbb{Z}$ is a surjective homomorphism of groups.

A *balanced set* is a subset $\mathbf{A} \subseteq \mathbf{D}$ such that, for every $\mathbf{a} \in \mathbf{A}$, there exist $\mathbf{b}_1, \mathbf{b}_2 \in \mathbf{A}$ such that $\mathbf{b}_1 \neq \mathbf{b}_2$ and $\mathbf{b}_1 + \mathbf{b}_2 = 2\mathbf{a}$. It is easy to see that the only finite balanced set is the empty set.

In the following, $\mathbb{A}$ will be a commutative integral domain with unity. The group of units of $\mathbb{A}$ will be denoted by $\mathbb{U}$.

A *series in* $\operatorname{card}(S)$ *variables* over $\mathbb{A}$ will be a map $\alpha: \mathbf{D} \to \mathbb{A}$. We shall frequently use the symbol $\langle \mathbf{d} | \alpha \rangle$ instead of $\alpha(\mathbf{d})$.
The series α will be written as

$$\alpha = (a_{\mathbf{d}})$$

with $a_{\mathbf{d}} = \langle \mathbf{d} | \alpha \rangle$, or—equivalently—as the formal sum

$$\alpha = \sum_{\mathbf{d} \in \mathbf{D}} a_{\mathbf{d}} \boldsymbol{\tau}^{\mathbf{d}},$$

where $\boldsymbol{\tau}^{\mathbf{d}}$ is the series such that

$$\langle \mathbf{e} | \boldsymbol{\tau}^{\mathbf{d}} \rangle = \delta_{\mathbf{e},\mathbf{d}}.$$

The zero series will be denoted by ζ and the $\mathbb{A}$-module of all series will be denoted by $\mathbf{S}$.

A collection $\{\alpha_i\}$ of series will be called a *summable collection* if for every $\mathbf{d} \in \mathbf{D}$ there exists only a finite number of indexes i such that $\langle \mathbf{d} | \alpha_i \rangle \neq 0$. If $\{\alpha_i\}$ is a summable collection, then the sum

$$\sum_i \alpha_i := \sum_{\mathbf{d}} \left(\sum_i \langle \mathbf{d} | \alpha_i \rangle \right) \boldsymbol{\tau}^{\mathbf{d}}$$

is well defined.

The notion of summability and sum of a summable collection is clearly equivalent to endow the $\mathbb{A}$-module $\mathbf{S}$ of the pointwise convergence topology with respect to the discrete topology over $\mathbb{A}$. Under this topology, $\mathbf{S}$ is a complete topological module.

A *Laurent series* over $\mathbb{A}$ is meant to be a series α whose support $\operatorname{supp} \alpha$ admits a lower bound in $\mathbf{D}$. In particular, a *power series* over $\mathbb{A}$ will be a series whose support is contained in $\mathbf{D}^+$. The sum of a summable collection of power series is a power series, while the same is false for Laurent series.

The *weight* of a Laurent series α is defined as

$$w(\alpha) := \min \{ w(\mathbf{d}),\ \mathbf{d} \in \operatorname{supp} \alpha \}.$$

By convention, $\min \varnothing = +\infty$ and $w(\zeta) = +\infty$.

A Laurent series α will be said to be a *homogeneous series* whenever all of its "monomials" have the same weight. The *leading homogeneous part* of the nonzero Laurent series

$$\alpha = \sum a_{\mathbf{d}} \boldsymbol{\tau}^{\mathbf{d}}$$

is the homogeneous series

$$\sum_{w(\mathbf{d}) = w(\alpha)} a_{\mathbf{d}} \boldsymbol{\tau}^{\mathbf{d}}.$$

The *linear part* of α is the homogeneous series

$$L(\alpha) = \sum_{w(\mathbf{d}) = 1} a_{\mathbf{d}} \boldsymbol{\tau}^{\mathbf{d}}.$$

The $\mathbb{A}$-modules $\mathbf{L}$ and $\mathbf{P}$ of all Laurent and power series over $\mathbb{A}$, respectively, can be endowed with the usual convolution product

$$\left(\sum_{\mathbf{d}} a_{\mathbf{d}} \boldsymbol{\tau}^{\mathbf{d}} \right) \left(\sum_{\mathbf{d}} b_{\mathbf{d}} \boldsymbol{\tau}^{\mathbf{d}} \right) := \sum_{\mathbf{d}} \left(\sum_{\mathbf{h}} a_{\mathbf{h}} b_{\mathbf{d}-\mathbf{h}} \right) \boldsymbol{\tau}^{\mathbf{d}}$$

and they turn out to be topological algebras, whose unity will be denoted by υ. The set $\{\tau_s; s \in S\}$, where

$$\begin{aligned} \langle \mathbf{d} | \tau_s \rangle &:= 1 \qquad \text{if} \quad \mathbf{d} = \mathbf{e}_s, \\ &:= 0 \qquad \text{otherwise}, \end{aligned}$$

is a set of pseudogenerators for $\mathbf{P}$ and $\mathbf{L}$; $\{\boldsymbol{\tau}^{\mathbf{d}}; \mathbf{d} \in \mathbf{D}^+\}$ and $\{\boldsymbol{\tau}^{\mathbf{d}}; \mathbf{d} \in \mathbf{D}\}$ are pseudobases for $\mathbf{P}$ and $\mathbf{L}$, respectively; and $\mathbf{P}$ is complete, while $\mathbf{L}$ is not complete. Furthermore, we have

(2.1) PROPOSITION. *$\mathbf{L}$ and $\mathbf{P}$ are integral domains.*

Proof. Let $\alpha, \beta \in \mathbf{P}$ and let α_0, β_0 be the leading homogeneous parts of α and β, respectively. Set $\mathbf{A} := \operatorname{supp} \alpha_0$ and $\mathbf{B} := \operatorname{supp} \beta_0$; then $\operatorname{supp} \alpha_0\beta_0 \subseteq \mathbf{A} + \mathbf{B}$. Suppose $\alpha\beta = \zeta$; then $\alpha_0\beta_0 = \zeta$ and $\operatorname{supp} \alpha_0\beta_0 = \varnothing$.

Let $\mathbf{d} \in \mathbf{A} + \mathbf{B}$; then $\mathbf{d}$ can be written in at least two different ways, namely,

$$\begin{aligned} \mathbf{d} &= \mathbf{a}_1 + \mathbf{b}_1, \qquad \mathbf{a}_1 \in \mathbf{A}, \mathbf{b}_1 \in \mathbf{B}, \\ \mathbf{d} &= \mathbf{a}_2 + \mathbf{b}_2, \qquad \mathbf{a}_2 \in \mathbf{A}, \mathbf{b}_2 \in \mathbf{B}, \end{aligned}$$

where $\mathbf{a}_1 \neq \mathbf{a}_2$, $\mathbf{b}_1 \neq \mathbf{b}_2$. The degrees

$$\mathbf{d}_1 := \mathbf{a}_1 + \mathbf{b}_2 \in \mathbf{A} + \mathbf{B}, \qquad \mathbf{d}_2 := \mathbf{a}_2 + \mathbf{b}_1 \in \mathbf{A} + \mathbf{B}$$

satisfy

$$2\mathbf{d} = \mathbf{d}_1 + \mathbf{d}_2$$

and

$$\operatorname{supp}(\mathbf{d}_i) \subseteq \operatorname{supp}(2\mathbf{d}), \qquad i = 1, 2.$$

Set $\mathbf{D} = \{\mathbf{e} \in \mathbf{A} + \mathbf{B}; \operatorname{supp}(\mathbf{e}) \subseteq \operatorname{supp}(\mathbf{d})\}$; then $\mathbf{D}$ is a balanced nonempty set, thus $\mathbf{D}$ is infinite. On the other hand $\mathbf{D}$ must be finite, because all degrees in $\mathbf{A} + \mathbf{B}$ have the same weight. By this contradiction, we conclude $\mathbf{A} + \mathbf{B} = \varnothing$, and hence $\mathbf{A} = \varnothing$ or $\mathbf{B} = \varnothing$, that is $\alpha = \zeta$ or $\beta = \zeta$. ∎

A Laurent series (or a power series) will be said to be a *principal series* whenever its support has a minimum. In this case, the *degree* of α is defined as

$$\deg \alpha := \min \operatorname{supp} \alpha.$$

By convention, $\deg \zeta = +\infty$.

The following is an easy consequence of Proposition 2.1:

(2.2) PROPOSITION. *Let α, β be Laurent (or power) series; then*

$$w(\alpha\beta) = w(\alpha) + w(\beta).$$

If α and β are both principal series, then $\alpha\beta$ is a principal series, and

$$\deg \alpha\beta = \deg \alpha + \deg \beta.$$

A series $\alpha \in \mathbf{P}$ has a multiplicative inverse whenever $\langle \mathbf{0} | \alpha \rangle \in \mathbb{U}$. In the following propositions we characterize those elements of $\mathbf{L}$ which admit multiplicative inverse.

(2.3) PROPOSITION. *Let α, β be homogeneous Laurent series such that $\alpha\beta = \boldsymbol{\tau}^{\mathbf{d}}$ for some $\mathbf{d} \in \mathbf{D}$; then there exist $a \in \mathbb{U}$ and $\mathbf{e} \in \mathbf{D}$ such that $\alpha = a\boldsymbol{\tau}^{\mathbf{e}}$ and $\beta = a^{-1}\boldsymbol{\tau}^{d-e}$.*

Proof. Without loss of generality, we can suppose that every monomial in α and β has nonnegative degree. Set $\mathbf{A} := \operatorname{supp} \alpha$ and $\mathbf{B} := \operatorname{supp} \beta$. Obviously, $\mathbf{d} \in \mathbf{A} + \mathbf{B}$. Suppose there exists $\mathbf{f} \in \mathbf{A} + \mathbf{B}$, $\mathbf{f} \neq \mathbf{d}$, and set

$$\mathbf{F} := \{\mathbf{g} \in \mathbf{A} + \mathbf{B};\ \operatorname{supp}(\mathbf{g}) \subseteq \operatorname{supp}(\mathbf{f})\}.$$

By the same argument used in the proof of Proposition 2.1, it can be shown that $\mathbf{F}$ is a nonempty balanced set and $\mathbf{F}$ is finite. Thus, $\mathbf{A} + \mathbf{B} = \{\mathbf{d}\}$ and the assertion follows. ∎

(2.4) PROPOSITION. *A Laurent series α has multiplicative inverse whenever it is a principal series, with $a_{\mathbf{d}} \in \mathbb{U}$, where $\mathbf{d} = \deg \alpha$.*

Proof. Suppose α is a principal series, with $a_{\mathbf{d}} \in \mathbb{U}$, $\mathbf{d} = \deg \alpha$; then, $\alpha = a_{\mathbf{d}}\boldsymbol{\tau}^{\mathbf{d}}(\upsilon + \hat{\alpha})$, where $\hat{\alpha}$ is a power series of positive weight. The collection

$$\{(-1)^h\, \hat{\alpha}^h;\ h \in \mathbb{N}\}$$

is summable, and the Laurent series

$$\beta := a_{\mathbf{d}}^{-1}\boldsymbol{\tau}^{-\mathbf{d}} \sum_{h \geq 0} (-1)^h\, \hat{\alpha}^h$$

is the multiplicative inverse of α. In order to prove the converse, it is sufficient to show that, if $\alpha, \beta \in \mathbf{P}$ and $\alpha\beta = \boldsymbol{\tau}^{\mathbf{d}}$, then α and β are principal series; this follows, in the usual way, from the isomorphism between $\mathbf{P}$ and the ring of power series in the variables $\{\tau_t;\ t \in T \subset S\}$ with coefficients taken in the ring of power series in the variables $\{\tau_s;\ s \in S - T\}$. ∎

The group of units of $\mathbf{L}$ will be denoted by $\mathbf{R}$. A summable collection $\boldsymbol{\alpha} = (\alpha_s)$, $s \in S$, of Laurent series will be called an **S**-*set*.

An **S**-set (α_s) will be called a **P**-*set* whenever

(i) $\alpha_s \in \mathbf{P}$ for every $s \in S$,

(ii) $w(\alpha_s) > 0$ for every $s \in S$.

An **S**-set (α_s) will be called an **L**-*set* whenever

(i) $\alpha_s \in \mathbf{R}$ for every $s \in S$,

(ii) $\deg(\alpha_s) > 0$ for every $s \in S$.

Given a **P**-set $\boldsymbol{\alpha} = (\alpha_s)$, $s \in S$, and a positive degree $\mathbf{d} = (d_s)$, the product

$$\boldsymbol{\alpha}^{\mathbf{d}} := \prod_s \alpha_s^{d_s}$$

is well-defined. We have

(2.5) PROPOSITION. *Let* $\boldsymbol{\alpha} = (\alpha_s)$ *be a* **P**-*set*; *then the collection* $\{\boldsymbol{\alpha}^{\mathbf{d}}; \mathbf{d} \in \mathbf{D}^+\}$ *is summable.*

Proof. Let $\mathbf{g} \in \mathbf{D}^+$ and set

$$\mathbf{G} := \{\mathbf{h} \in \mathbf{D}^+; \mathbf{0} \leqslant \mathbf{h} \leqslant \mathbf{g}\}, \qquad T := \{s \in S; \operatorname{supp} \alpha_s \cap \mathbf{G} \neq \varnothing\};$$

$\mathbf{G}$ and T are finite sets. Let now $\mathbf{f} \in \mathbf{D}^+$ be such that $\mathbf{g} \in \operatorname{supp} \boldsymbol{\alpha}^{\mathbf{f}}$; then, for every $s \in \operatorname{supp}(\mathbf{f})$, there exists $\mathbf{e}_{s,i} \in \operatorname{supp} \alpha_s$, $1 \leqslant i \leqslant \mathbf{f}(s)$, such that

$$\mathbf{g} = \sum_s \sum_{i=1}^{f(s)} \mathbf{e}_{s,i} \qquad \mathbf{e}_{s,i} > \mathbf{0}, \quad \mathbf{e}_{s,i} \in \mathbf{G}.$$

This implies that $\operatorname{supp}(\mathbf{f}) \subseteq T$ and $w(\mathbf{f}) \leqslant w(\mathbf{g})$: henceworth, $\mathbf{f}$ can be chosen in a finite number of ways. ∎

Given an **L**-set $\boldsymbol{\alpha} = (\alpha_s)$, $s \in S$, the product

$$\boldsymbol{\alpha}^{\mathbf{d}} := \prod_s \alpha_s^{d_s}$$

is well-defined for every degree $\mathbf{d} \in \mathbf{D}$. We have

(2.6) PROPOSITION. *Let* $\boldsymbol{\alpha} = (\alpha_s)$ *be an* **L**-*set and* $\mathbf{d} \in \mathbf{D}$; *then, the collection* $\{\boldsymbol{\alpha}^{\mathbf{g}}; \mathbf{g} \geqslant \mathbf{d}\}$ *is summable.*

Proof. If $\mathbf{d} \geqslant \mathbf{0}$, the proposition follows by 2.5, since every **L**-set is a **P**-set. Suppose $\mathbf{d} \ngeqslant \mathbf{0}$; then $\{\boldsymbol{\alpha}^{\mathbf{g}}; \mathbf{g} \geqslant \mathbf{d}\} = \{\boldsymbol{\alpha}^{\mathbf{f}}\boldsymbol{\alpha}^{\mathbf{d}}; \mathbf{f} \geqslant \mathbf{0}\} = \boldsymbol{\alpha}^{\mathbf{d}}\{\boldsymbol{\alpha}^{\mathbf{f}}; \mathbf{f} \geqslant 0\}$, and $\{\boldsymbol{\alpha}^{\mathbf{g}}; \mathbf{g} \geqslant \mathbf{d}\}$ is summable. ∎

By the preceding propositions, given any **P**-set $\boldsymbol{\alpha}$ and a power series

$\beta = \sum b_{\mathbf{d}} \boldsymbol{\tau}^{\mathbf{d}}$ (or any **L**-set and a Laurent series, respectively), we can define the *composition* $\beta \circ \boldsymbol{\alpha}$ as

$$\beta \circ \boldsymbol{\alpha} := \sum_{\mathbf{d} \in \operatorname{supp} \beta} b_{\mathbf{d}} \boldsymbol{\alpha}^{d}.$$

We have

(2.7) PROPOSITION. *Let $\boldsymbol{\alpha} = (\alpha_s)$ be a* **P**-*set and β be a power series (or an* **L**-*set and a Laurent series, respectively); then, $\beta \circ \boldsymbol{\alpha}$ is a power series (a Laurent series). Moreover,*

$$w(\beta \circ \boldsymbol{\alpha}) = \inf \left\{ \sum_{s \in \operatorname{supp}(\mathbf{g})} \mathbf{g}(s)\, w(\alpha_s); \mathbf{g} \in \operatorname{supp} \beta \right\}.$$

If β is principal series, with $\mathbf{b} = \deg(\beta)$, and if α_s is a principal series for every $s \in S$, with $\mathbf{a}_s = \deg(\alpha_s)$, then $\beta \circ \boldsymbol{\alpha}$ is a principal series, with

$$\deg(\beta \circ \boldsymbol{\alpha}) = \sum_{s \in \operatorname{supp}(\mathbf{b})} b_s \mathbf{a}_s.$$

Let $\boldsymbol{\alpha} = (\alpha_s)$ and $\boldsymbol{\beta} = (\beta_s)$ be **P**-sets (**L**-sets). The composition $\boldsymbol{\alpha} \circ \boldsymbol{\beta}$ is defined as the **S**-set $\boldsymbol{\alpha} \circ \boldsymbol{\beta} := (\alpha_s \circ \boldsymbol{\beta})$.

(2.8) PROPOSITION. *Let $\boldsymbol{\alpha}, \boldsymbol{\beta}$ be both* **P**-*sets* (**L**-sets): *then $\boldsymbol{\alpha} \circ \boldsymbol{\beta}$ is a* **P**-*set* (**L**-*set*).

Proof. Set $\gamma_s := \alpha_s \circ \boldsymbol{\beta}$ and $\boldsymbol{\gamma} := (\gamma_s)$, $s \in S$. By Proposition 2.8, we have only to show that $\boldsymbol{\gamma}$ is summable, whenever $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are both **P**-sets. For every $\mathbf{f} \in \mathbf{D}^+$, set

$$A(\mathbf{f}) := \{s \in S; \mathbf{f} \in \operatorname{supp} \beta_s\}$$

and

$$B(\mathbf{f}) := \bigcup_{\mathbf{g} < \mathbf{f}} A(\mathbf{g}).$$

Then $A(\mathbf{f})$ is finite, since $\boldsymbol{\beta}$ is summable, and $B(\mathbf{f})$ is finite, since it is a finite union of finite sets. Setting now

$$C(\mathbf{f}) := \{s \in S;\ \mathbf{f} \in \operatorname{supp} \gamma_s\}$$

we have $C(\mathbf{f}) \subseteq B(\mathbf{f})$. Hence, $\boldsymbol{\gamma}$ is summable. ∎

The families of all **P**-sets and of all **L**-sets, endowed with the composition defined above, turn out to be monoids, whose identity is $\boldsymbol{\tau}$.

If $\boldsymbol{\alpha} = (\alpha_s)$ and $\boldsymbol{\beta} = (\beta_s)$ are **S**-sets, the *sum* $\boldsymbol{\alpha} + \boldsymbol{\beta}$ is defined pointwise

$$\boldsymbol{\alpha} + \boldsymbol{\beta} = (\alpha_s + \beta_s).$$

If $\boldsymbol{\alpha} \circ \boldsymbol{\gamma}$, $\boldsymbol{\beta} \circ \boldsymbol{\gamma}$, $(\boldsymbol{\alpha} + \boldsymbol{\beta}) \circ \boldsymbol{\gamma}$ are defined, then

$$(\boldsymbol{\alpha} + \boldsymbol{\beta}) \circ \boldsymbol{\gamma} = \boldsymbol{\alpha} \circ \boldsymbol{\gamma} + \boldsymbol{\beta} \circ \boldsymbol{\gamma}.$$

The *linear part* of an **S**-set $\boldsymbol{\alpha} = (\alpha_s)$ is the **P**-set $L(\boldsymbol{\alpha}) = (L(\alpha_s))$ of all linear parts of the series α_s. An **S**-set $\boldsymbol{\alpha}$ will be called *linear* whenever $L(\boldsymbol{\alpha}) = \boldsymbol{\alpha}$.

If $\boldsymbol{\alpha}$ is a linear **S**-set and $\boldsymbol{\alpha} \circ \boldsymbol{\beta}$, $\boldsymbol{\alpha} \circ \boldsymbol{\gamma}$, $\boldsymbol{\alpha} \circ (\boldsymbol{\beta} + \boldsymbol{\gamma})$ are defined, then

$$\boldsymbol{\alpha} \circ (\boldsymbol{\beta} + \boldsymbol{\gamma}) = \boldsymbol{\alpha} \circ \boldsymbol{\beta} + \boldsymbol{\alpha} \circ \boldsymbol{\gamma}.$$

If $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are **P**-sets (**L**-sets), then

$$L(\boldsymbol{\alpha} \circ \boldsymbol{\beta}) = L(\boldsymbol{\alpha}) \circ L(\boldsymbol{\beta}).$$

In order to characterize **P**-sets and **L**-sets admitting two-sided inverse, we need

(2.9) PROPOSITION. *Every* **P**-*set* (**L**-*set*) $\boldsymbol{\alpha}$ *is right regular, that is,* $\boldsymbol{\beta} \circ \boldsymbol{\alpha} = \boldsymbol{\alpha}$ *implies* $\boldsymbol{\beta} = \boldsymbol{\tau}$.

(2.10) PROPOSITION. *Let* $\boldsymbol{\alpha} = (\alpha_s)$ *be a* **P**-*set* (**L**-*set*) *such that* $w(\alpha_s) \geqslant 2$ *for every* $s \in S$, *and let* $\boldsymbol{\phi}$ *be an invertible linear* **P**-*set* (**L**-*set*); *then, there exists exactly one* **P**-*set* $\boldsymbol{\beta} := (\beta_s)$ *such that*

$$\boldsymbol{\alpha} \circ (\boldsymbol{\phi} + \boldsymbol{\beta}) = -\boldsymbol{\beta}. \tag{$*$}$$

Moreover, we have $w(\beta_s) \geqslant 2$ *for every* $s \in S$.

Proof. By Proposition 2.7, if there exists $\boldsymbol{\beta}$ such that $\boldsymbol{\alpha} \circ (\boldsymbol{\phi} + \boldsymbol{\beta}) = -\boldsymbol{\beta}$, then $w(\beta_s) \geqslant w(\alpha_s) \geqslant 2$. For every $n \geqslant 2$, set

$$\boldsymbol{\beta}_n = (\beta_{s,n}),$$

where

$$\beta_{s,n} = \sum_{w(\mathbf{d}) = n} \langle \mathbf{d} | \beta_s \rangle \, \boldsymbol{\tau}^{\mathbf{d}}.$$

Hence, Eq. $(*)$ is equivalent to the system

$$\langle \mathbf{d} | \alpha_s \circ (\boldsymbol{\phi} + \boldsymbol{\beta}) \rangle = -\langle \mathbf{d} | \beta_s \rangle, \qquad s \in S, \mathbf{d} \in \mathbf{D}^+. \tag{$**$}$$

On the other hand, $w(\alpha_s) \geqslant 2$ implies that

$$\langle \mathbf{d} | \alpha_s \circ (\boldsymbol{\phi} + \boldsymbol{\beta}) \rangle = \left\langle \mathbf{d} | \alpha_s \circ \left(\boldsymbol{\phi} + \sum_{i=2}^{w(\mathbf{d})-1} \boldsymbol{\beta}_i \right) \right\rangle,$$

and the system $(**)$ has the unique solution recursively given by

$$\langle \mathbf{d} | \beta_s \rangle = \left\langle \mathbf{d} | \alpha_s \circ \left(\boldsymbol{\phi} + \sum_{i=2}^{w(\mathbf{d})-1} \boldsymbol{\beta}_i \right) \right\rangle. \quad \blacksquare$$

We have

(2.11) THEOREM. *A* **P**-*set* (**L**-set) $\boldsymbol{\alpha} = (\alpha_s)$ *admits two-sided inverse if and only if its linear part* $L(\boldsymbol{\alpha})$ *is an invertible* **P**-*set* (**L**-*set*).

Proof. Suppose $\boldsymbol{\alpha}$ admits two-sided inverse $\boldsymbol{\beta}$: then,

$$L(\boldsymbol{\alpha}) \circ L(\boldsymbol{\beta}) = L(\boldsymbol{\alpha} \circ \boldsymbol{\beta}) = L(\boldsymbol{\tau}) = \boldsymbol{\tau},$$

and

$$L(\boldsymbol{\beta}) \circ L(\boldsymbol{\alpha}) = L(\boldsymbol{\beta} \circ \boldsymbol{\alpha}) = L(\boldsymbol{\tau}) = \boldsymbol{\tau};$$

hence, $L(\boldsymbol{\alpha})$ is invertible.

Conversely, assume $L(\boldsymbol{\alpha})$ invertible and let $\tilde{L}(\boldsymbol{\alpha})$ be its inverse. Set $\boldsymbol{\alpha} = L(\boldsymbol{\alpha}) + M(\boldsymbol{\alpha})$. By Proposition 2.10, there exists exactly one set $\boldsymbol{\gamma} = (\gamma_s)$ such that $w(\gamma_s) \geqslant 2$ for every $s \in S$ and $(\tilde{L}(\boldsymbol{\alpha}) \circ M(\boldsymbol{\alpha})) \circ (\tilde{L}(\boldsymbol{\alpha}) + \boldsymbol{\gamma}) = -\boldsymbol{\gamma}$. Setting $\boldsymbol{\beta} = \tilde{L}(\boldsymbol{\alpha}) + \boldsymbol{\gamma}$, we have $\boldsymbol{\alpha} \circ \boldsymbol{\beta} = \boldsymbol{\tau}$. Moreover, $(\boldsymbol{\beta} \circ \boldsymbol{\alpha}) \circ \boldsymbol{\beta} = \boldsymbol{\beta} \circ (\boldsymbol{\alpha} \circ \boldsymbol{\beta}) = \boldsymbol{\beta}$ implies, by Proposition 2.9, that $\boldsymbol{\beta} \circ \boldsymbol{\alpha} = \boldsymbol{\tau}$. This shows that $\boldsymbol{\beta}$ is the two-sided inverse of $\boldsymbol{\alpha}$. $\blacksquare$

The two-sided inverse of an invertible set $\boldsymbol{\alpha}$ will be denoted by $\tilde{\boldsymbol{\alpha}}$. The preceding theorem allows us to characterize the invertible **L**-sets as those **L**-sets $\boldsymbol{\alpha} = (\alpha_s)$ such that $\deg(\alpha_s) = \mathbf{e}_{\sigma(s)}$ for some permutation σ of S. The bijection σ will be called the *spire* of $\boldsymbol{\alpha}$, and we will write $\mathrm{sp}(\boldsymbol{\alpha}) := \sigma$. An invertible **L**-set $\boldsymbol{\alpha}$ will be called *normalized* if $\mathrm{sp}(\boldsymbol{\alpha})$ is the identity permutation i on S. The *normalization* of an invertible **L**-set $\boldsymbol{\alpha} := (\alpha_s)$ is the invertible **L**-set $\boldsymbol{\beta} := (\beta_s)$, where $\beta_s := \alpha_{\sigma^{-1}(s)}$.

If σ is a bijection of S, we set

$$\boldsymbol{\tau}^{\sigma} := (\tau_{\sigma(s)}).$$

Given a **P**-set (**L**-set) $\boldsymbol{\alpha}$, the map $\beta \to \beta \circ \boldsymbol{\alpha}$ is a continuous endomorphism of **P** (of **L**). Furthermore, if ψ is an endomorphism of $\mathbb{A}$ and $\alpha = (a_{\mathbf{d}})$ is a series, we set $\alpha^{\psi} = (a_{\mathbf{d}}^{\psi})$; then, $\psi: \alpha \to \alpha^{\psi}$ is a continuous endomorphism of **P** (of **L**).

(2.12) PROPOSITION. *For every nonzero continuous endomorphism Ψ of* **P** *(of* **L***), there exist exactly one endomorphism ψ of $\mathbb{A}$ and one* **P***-set (***L***-set)* $\boldsymbol{\alpha}$ *such that, for every $\beta \in \mathbf{P}$ ($\beta \in \mathbf{L}$), $\Psi(\beta) = \beta^{\psi} \circ \boldsymbol{\alpha}$.*

Proof. For every $a \in \mathbb{A}$, set $a^{\psi} = \Psi(a)$. Then Ψ is an endomorphism of $\mathbb{A}$. Now, set $\alpha_s = \Psi(\tau_s)$ and $\boldsymbol{\alpha} = (\alpha_s)$. Then $\boldsymbol{\alpha}$ is a **P**-set (an **L**-set) and $\Psi(\beta) = \beta^{\psi} \circ \boldsymbol{\alpha}$ for every $\beta \in \mathbf{P}$ ($\beta \in \mathbf{L}$). ∎

The endomorphism $\psi \in \mathrm{End}(A)$ will be called the *companion endomorphism* of Ψ. A *linear endomorphism* of **P** (of **L**) will be an endomorphism whose companion is the identity on $\mathbb{A}$.

For any given normalized invertible **L**-set $\boldsymbol{\alpha} = (\alpha_s)$, consider the series

$$\begin{aligned} \frac{\tau_i}{\alpha_j}\frac{\partial \alpha_j}{\partial \tau_i} &= 1 + \frac{\tau_i}{\hat{\alpha}_i}\frac{\partial \hat{\alpha}_i}{\partial \tau_i} && \text{if} \quad i = j, \\ &= \frac{\tau_i}{\hat{\alpha}_j}\frac{\partial \hat{\alpha}_j}{\partial \tau_i} && \text{if} \quad i \neq j, \end{aligned}$$

where $\partial \alpha_j / \partial \tau_i$ denotes the usual partial derivative, and $\alpha_i = \tau_i \hat{\alpha}_i$. By straightforward computations we get

$$\left\langle \mathbf{0} \,\middle|\, \frac{\tau_i}{\alpha_j}\frac{\partial \alpha_j}{\partial \tau_i} \right\rangle = \delta_{ij};$$

hence, the degree of the power series $(\tau_i/\alpha_j)(\partial \alpha_j/\partial \tau_i)$ is zero if and only if $i = j$. Moreover, if $i \neq j$, then

$$\mathbf{d} \in \operatorname{supp} \frac{\tau_i}{\alpha_j}\frac{\partial \alpha_j}{\partial \tau_i} \qquad \text{implies} \quad i \in \operatorname{supp}(\mathbf{d}).$$

For every finite subset T of S, we set

$$P_T(\boldsymbol{\alpha}) = \det \left(\frac{\tau_i}{\alpha_j}\frac{\partial \alpha_j}{\partial \tau_i} \right)_{i,j \in T} = \det \left(I + \left(\frac{\tau_i}{\hat{\alpha}_j}\frac{\partial \hat{\alpha}_j}{\partial \tau_i} \right) \right).$$

By the preceding remarks, we have

(2.13) PROPOSITION. *Let $\boldsymbol{\alpha} = (\alpha_s)$ be a normalized invertible* **L***-set and let $\mathbf{d} \in \mathbf{D}^+$, $T = \operatorname{supp}(\mathbf{d})$. Let U be a finite subset of S; if $T \subseteq U$, then*

$$\langle \mathbf{d} | P_U(\boldsymbol{\alpha}) \rangle = \langle \mathbf{d} | P_T(\boldsymbol{\alpha}) \rangle.$$

If S is finite, we set

$$P(\boldsymbol{\alpha}) = P_S(\boldsymbol{\alpha}).$$

If S is not finite, then the sequence $P_T(\boldsymbol{\alpha})$, where T ranges over the filter of all finite subset of S, converges to the series $P(\boldsymbol{\alpha})$ defined as follows: for every $\mathbf{d} \in \mathbf{D}^+$,

$$\langle \mathbf{d} | P(\boldsymbol{\alpha}) \rangle := \langle \mathbf{d} | P_{\operatorname{supp}(\mathbf{d})}(\boldsymbol{\alpha}) \rangle.$$

For any given invertible $\mathbf{L}$-set $\boldsymbol{\alpha}$, we set

$$P(\boldsymbol{\alpha}) := P(\boldsymbol{\alpha}'),$$

where $\boldsymbol{\alpha}'$ is the normalization of $\boldsymbol{\alpha}$.

(2.14) PROPOSITION. *Let $\boldsymbol{\alpha} = (\alpha_s)$ and $\boldsymbol{\beta} = (\beta_s)$ be invertible* $\mathbf{L}$*-sets; then*

$$P(\boldsymbol{\alpha} \circ \boldsymbol{\beta}) = (P(\boldsymbol{\alpha}) \circ \boldsymbol{\beta})\, P(\boldsymbol{\beta})$$

and

$$P(\tilde{\boldsymbol{\alpha}}) = (P(\boldsymbol{\alpha}))^{-1} \circ \boldsymbol{\alpha}.$$

Proof. By the definition of $P(\boldsymbol{\alpha})$, it is sufficient to prove the identities above in the case when $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are normalized and S is finite. We have

$$\begin{aligned} P(\boldsymbol{\alpha} \circ \boldsymbol{\beta}) &= \det \left(\frac{\tau_i}{\alpha_j \circ \boldsymbol{\beta}} \frac{\partial(\alpha_j \circ \boldsymbol{\beta})}{\partial \tau_i} \right) \\ &= \det \left(\frac{\beta_i}{\alpha_j \circ \boldsymbol{\beta}} \left(\frac{\partial \alpha_j}{\partial \tau_i} \right) \circ \boldsymbol{\beta} \right) \det \left(\frac{\tau_i}{\beta_j} \frac{\partial \beta_j}{\partial \tau_i} \right) \\ &= (P(\boldsymbol{\alpha}) \circ \boldsymbol{\beta})\, P(\boldsymbol{\beta}). \end{aligned}$$

Moreover,

$$\upsilon = P(\boldsymbol{\tau}) = P(\boldsymbol{\alpha} \circ \tilde{\boldsymbol{\alpha}}) = (P(\boldsymbol{\alpha}) \circ \tilde{\boldsymbol{\alpha}})\, P(\tilde{\boldsymbol{\alpha}}),$$

which implies

$$P(\tilde{\boldsymbol{\alpha}}) = (P(\boldsymbol{\alpha}))^{-1} \circ \tilde{\boldsymbol{\alpha}}. \quad \blacksquare$$

(2.15) THEOREM. *Let $\boldsymbol{\alpha} = (\alpha_s)$ be an invertible* $\mathbf{L}$*-set; for every $\mathbf{d} \in \mathbf{D}$, we have*

$$\begin{aligned} \langle \mathbf{0} | \boldsymbol{\alpha}^{\mathbf{d}} P(\boldsymbol{\alpha}) \rangle &= 1 \qquad \text{if} \quad \mathbf{d} = \mathbf{0}, \\ &= 0 \qquad \text{otherwise.} \end{aligned}$$

Proof. In order to realize that the assertion holds in a characteristic free setting, it is sufficient to perform our computations in the case when $\mathbb{A}$ is a free commutative $\mathbb{Z}$-algebra with unity. We can suppose, without loss of generality, that $\boldsymbol{\alpha}$ is normalized. First of all, we remark that, for every $\mathbf{d} \in \mathbf{D}$, $\deg(\boldsymbol{\alpha}^{\mathbf{d}}) = \mathbf{d}$ and, hence, $\mathbf{d} \nless \mathbf{0}$ implies $\mathbf{0} \notin \operatorname{supp} \boldsymbol{\alpha}^{\mathbf{d}} P(\boldsymbol{\alpha})$. Furthermore, $\langle \mathbf{0} | P(\boldsymbol{\alpha}) \rangle = 1$.

This implies that the theorem is established by proving that, for every $\mathbf{d} \in \mathbf{D}$, $\mathbf{d} < \mathbf{0}$,

$$\langle \mathbf{0} | \|\mathbf{d}\| \, \boldsymbol{\alpha}^{\mathbf{d}} P_T(\boldsymbol{\alpha}) \rangle = 0,$$

where

$$T = \operatorname{supp}(\mathbf{d}) = \{1, 2, ..., n\}$$

in place of $\{i_1, i_2, ..., i_n\}$ and

$$\|\mathbf{d}\| = \prod_{i=1}^{n} \mathbf{d}(i).$$

We have

$$\begin{aligned}
\langle \mathbf{0} | \|\mathbf{d}\| \cdot \boldsymbol{\alpha}^{\mathbf{d}} P_T(\boldsymbol{\alpha}) \rangle &= \left\langle \mathbf{0} \,\middle|\, \det \left(\mathbf{d}(j) \, \boldsymbol{\alpha}^{\mathbf{d}(j)} \frac{\tau_i}{\alpha_j} \frac{\partial \alpha_j}{\partial \tau_i} \right) \right\rangle \\
&= \left\langle \mathbf{0} \,\middle|\, \det \left(\tau_i \frac{\partial \alpha_j^{\mathbf{d}(j)}}{\partial \tau_i} \right) \right\rangle \\
&= \sum_{\Sigma \mathbf{g}_j = \mathbf{0}} \det \left\langle \mathbf{g}_j \,\middle|\, \tau_i \frac{\partial \alpha_j^{\mathbf{d}(j)}}{\partial \tau_i} \right\rangle \\
&= \sum_{\Sigma \mathbf{g}_j = \mathbf{0}} \langle \mathbf{g}_j | \alpha_j^{\mathbf{d}(j)} \rangle \det(\mathbf{g}_j(i)).
\end{aligned}$$

But $\Sigma \mathbf{g}_j = \mathbf{0}$ implies $\det(\mathbf{g}_j(\mathrm{i})) = 0$ and the assertion is proved. ∎

3. Recursive Matrices

In the sequel, a *matrix* will be a map

$$M : \mathbf{D} \times \mathbf{D} \to \mathbb{A}.$$

When M is a matrix, we frequently write $m_{\mathbf{f},\mathbf{g}}$ for $M(\mathbf{f}, \mathbf{g})$ and $M = (m_{\mathbf{f},\mathbf{g}})$.

Given a degree $\mathbf{d} \in \mathbf{D}$, the $\mathbf{d}$th *row-generating function* of the matrix $M = (m_{\mathbf{f},\mathbf{g}})$ will be the series

$$\langle M | \mathbf{d} \rangle := \sum_{\mathbf{g} \in \mathbf{D}} m_{\mathbf{d},\mathbf{g}} \tau^{\mathbf{g}}.$$

We define an involutory linear operator T on the $\mathbb{A}$-module of all matrices as follows: if $M = (m_{\mathbf{f},\mathbf{g}})$, then $TM := (m_{-\mathbf{g},-\mathbf{f}})$.

A matrix will be said to be a *Laurent matrix* whenever all of its row-generating functions are Laurent series. A matrix M will be called an *inverse Laurent matrix* if TM is a Laurent matrix. A matrix $M = (m_{\mathbf{f},\mathbf{g}})$ will be said to be *diagonally finite* whenever, for every $\mathbf{h}, \mathbf{k} \in \mathbf{D}$ there exists only a finite number of nonzero entries $m_{\mathbf{f},\mathbf{g}}$ with $\mathbf{f} \geqslant \mathbf{h}$, $\mathbf{g} \leqslant \mathbf{k}$.

Every diagonally finite matrix is both a Laurent and an inverse Laurent matrix, while the converse is false.

Given two matrices $M = (m_{\mathbf{f},\mathbf{g}})$ and $N = (n_{\mathbf{f},\mathbf{g}})$, the usual *convolution product*

$$M \times N = (p_{\mathbf{f},\mathbf{g}})$$

with

$$p_{\mathbf{f},\mathbf{g}} := \sum_{\mathbf{k} \in \mathbf{D}} m_{\mathbf{f},\mathbf{k}} n_{\mathbf{k},\mathbf{g}}$$

is well defined whenever M is a Laurent matrix and N is an inverse Laurent matrix.

If the product $M \times N$ is defined, then

$$T(M \times N) = TN \times TM.$$

For every $s \in S$, we define two invertible linear operators F_s and G_s acting over the $\mathbb{A}$-module of all matrices, as follows: if $M = (m_{\mathbf{f},\mathbf{g}})$, we set

$$F_s M := (m_{\mathbf{f}+\mathbf{e}_s,\mathbf{g}})$$

and

$$G_s M := (m_{\mathbf{f},\mathbf{g}-\mathbf{e}_s}).$$

We remark that

$$G_s = TF_s T$$

and, for every s, $t \in S$,

$$F_s G_t = G_t F_s.$$

Moreover, for every matrix M and every degree $\mathbf{d} \in \mathbf{D}$,

$$\langle F_s M | \mathbf{f} \rangle = \langle M | \mathbf{f} + \mathbf{e}_s \rangle, \qquad \langle G_s M | \mathbf{f} \rangle = \tau_s \langle M | \mathbf{f} \rangle.$$

For every $\mathbf{d} \in \mathbf{D}$, we set

$$\mathbf{F}^{\mathbf{d}} := \prod_{s\in S} F_s^{d(s)} \qquad \text{and} \qquad \mathbf{G}^{\mathbf{d}} := \prod_{s\in S} G_s^{d(s)}.$$

Thus, for every Laurent series $\alpha = \sum_{\mathbf{d}\in\mathbf{D}} a_{\mathbf{d}}\boldsymbol{\tau}^{\mathbf{d}}$, the formal writings

$$\alpha(\mathbf{F}) := \sum_{\mathbf{d}\in\mathbf{D}} a_{\mathbf{d}}\mathbf{F}^{\mathbf{d}} \qquad \text{and} \qquad \alpha(\mathbf{G}) := \sum_{\mathbf{d}\in\mathbf{D}} a_{\mathbf{d}}\mathbf{G}^{\mathbf{d}}$$

define two linear operators over the $\mathbb{A}$-module of all inverse Laurent matrices and of all Laurent matrices, respectively.

If $\boldsymbol{\alpha} = (\alpha_s)$ is an $\mathbf{S}$-set, a $\mathbf{P}$-set, or an $\mathbf{L}$-set, the symbols $\boldsymbol{\alpha}(\mathbf{F})$ and $\boldsymbol{\alpha}(\mathbf{G})$ will denote the collections of operators $(\alpha_s(\mathbf{F}))$ and $(\alpha_s(\mathbf{G}))$, respectively.

We remark that, by definitions, for every $\mathbf{d} \in \mathbf{D}$ we have

$$\mathbf{G}^{\mathbf{d}} = T\mathbf{F}^{\mathbf{d}}T;$$

hence, for every $\mathbf{S}$-set $\boldsymbol{\alpha}$

$$\boldsymbol{\alpha}(\mathbf{G}) = T\boldsymbol{\alpha}(\mathbf{F})\,T.$$

On the $\mathbb{A}$-module of all Laurent matrices, we now consider the linear problem

$$\boldsymbol{\alpha}(\mathbf{G})\,M = \mathbf{F}M, \qquad \langle M|\mathbf{0}\rangle = \beta, \tag{$*$}$$

where $\boldsymbol{\alpha} = (\alpha_s)$ is an $\mathbf{S}$-set such that $\alpha_s \in \mathbf{R}$ for every $s \in S$ and $\beta \in \mathbf{L}$, $\beta \neq \zeta$. It is easily seen, by construction, that for every such $\mathbf{S}$-set $\boldsymbol{\alpha}$ and for every nonzero Laurent series β, the linear problem $(*)$ has unique solution. We get immediately

(3.1) Proposition. *Let $\boldsymbol{\alpha} = (\alpha_s)$ be an $\mathbf{S}$-set such that for every $s \in S$ $\alpha_s \in \mathbf{R}$, and let $\beta \in \mathbf{L}$, $\beta \neq \zeta$. Then the Laurent matrix M is the unique solution of the linear problem $(*)$ if and only if, for every $\mathbf{d} \in \mathbf{D}$,*

$$\langle M|\mathbf{d}\rangle = \boldsymbol{\alpha}^{\mathbf{d}}\beta.$$

As a consequence, we have

(3.2) Proposition (Convolution formula). *Let $\boldsymbol{\alpha} = (\alpha_s)$ be an $\mathbf{S}$-set, with $\alpha_s \in \mathbf{R}$ for every $s \in S$, and let $\beta \in \mathbf{L}$, $\beta \neq \zeta$. The Laurent matrix $M = (m_{\mathbf{f},\mathbf{g}})$ is the unique solution of the linear problem $(*)$ if and only if $\langle M|\mathbf{0}\rangle = \beta$ and for every $\mathbf{f}, \mathbf{g}, \mathbf{h} \in \mathbf{D}$ the following identities hold:*

$$m_{\mathbf{f}+\mathbf{h},\mathbf{g}} = \sum_{\mathbf{d}\in\mathbf{D}} c_{\mathbf{f},\mathbf{d}}\, m_{\mathbf{h},\mathbf{g}-\mathbf{d}},$$

where $c_{\mathbf{f},\mathbf{d}} = \langle \mathbf{d}|\boldsymbol{\alpha}^{\mathbf{f}}\rangle$.

Proposition 3.1 suggests that we call *$(\boldsymbol{\alpha}, \beta)$-recursive matrix*, $R(\boldsymbol{\alpha}, \beta)$ for short, the unique solution of problem $(*)$. $\boldsymbol{\alpha}$ will be called the *recurrence rule* and β the *boundary value* of $R(\boldsymbol{\alpha}, \beta)$.

A *homogeneous matrix* will be a recursive matrix with boundary value υ, and an *Appell matrix* will be a recursive matrix with recurrence rule $\boldsymbol{\tau}$. The *identity matrix* $R(\boldsymbol{\tau}, \upsilon)$ is the unique Appell homogeneous matrix. For any given permutation σ on S, the *σ-permutation matrix* will be the recursive matrix $R(\boldsymbol{\tau}^\sigma, \upsilon)$, where $\boldsymbol{\tau}^\sigma = (\tau_{\sigma(s)})$.

We remark that a recursive matrix $R(\boldsymbol{\alpha}, \beta)$ is a diagonally finite matrix if and only if for every $s \in S$: $\deg \alpha_s > 0$, that is, if and only if $\boldsymbol{\alpha}$ is an $\mathbf{L}$-set. It follows that the product of two recursive matrices is defined whenever the recurrence rule of the second one is an $\mathbf{L}$-set. Furthermore, we have

(3.3) THEOREM. *Let $\boldsymbol{\alpha} = (\alpha_s)$ be an $\mathbf{S}$-set such that $\alpha_s \in \mathbf{R}$ for every $s \in S$. Let $\boldsymbol{\gamma}$ be an $\mathbf{L}$-set and let $\beta, \delta \in \mathbf{L}$, $\beta \neq \zeta \neq \delta$. Then*

$$R(\boldsymbol{\alpha}, \beta) \times R(\boldsymbol{\gamma}, \delta) = R(\boldsymbol{\alpha} \circ \boldsymbol{\gamma}, (\beta \circ \boldsymbol{\gamma})\, \delta).$$

Proof. Set $P := (p_{\mathbf{f},\mathbf{g}}) := R(\boldsymbol{\alpha}, \beta)$, $Q := (q_{\mathbf{f},\mathbf{g}}) := R(\boldsymbol{\gamma}, \delta)$ and $M := (m_{\mathbf{f},\mathbf{g}}) := P \times Q$. We have:

$$\begin{aligned}
\langle M | \mathbf{d} \rangle &= \sum_{\mathbf{g} \in \mathbf{D}} m_{\mathbf{d},\mathbf{g}} \boldsymbol{\tau}^{\mathbf{d}} = \sum_{\mathbf{g} \in \mathbf{D}} \sum_{\mathbf{f} \in \mathbf{D}} p_{\mathbf{d},\mathbf{f}} q_{\mathbf{f},\mathbf{g}} \boldsymbol{\tau}^{\mathbf{g}} \\
&= \sum_{\mathbf{f} \in \mathbf{D}} p_{\mathbf{d},\mathbf{f}} \langle Q | \mathbf{f} \rangle \\
&= \sum_{\mathbf{f} \in \mathbf{D}} p_{\mathbf{d},\mathbf{f}} \boldsymbol{\gamma}^{\mathbf{f}} \delta = \left(\left(\sum_{\mathbf{f} \in \mathbf{D}} p_{\mathbf{d},\mathbf{f}} \boldsymbol{\tau}^{\mathbf{f}} \right) \circ \boldsymbol{\gamma} \right) \delta \\
&= ((\boldsymbol{\alpha}^{\mathbf{d}} \beta) \circ \boldsymbol{\gamma})\, \delta = (\boldsymbol{\alpha} \circ \boldsymbol{\gamma})^{\mathbf{d}} (\beta \circ \boldsymbol{\gamma})\, \delta.
\end{aligned}$$

By Proposition 3.1 we get the assertion. ∎

(3.4) COROLLARY. *Let $\boldsymbol{\alpha} = (\alpha_s)$ be an $\mathbf{S}$-set with $\alpha_s \in \mathbf{R}$ for every $s \in S$, and let $\beta \in \mathbf{L}$, $\beta \neq \zeta$. Then*

$$R(\boldsymbol{\alpha}, \beta) = R(\boldsymbol{\alpha}, \upsilon) \times R(\boldsymbol{\tau}, \beta).$$

(3.5) COROLLARY. *A recursive matrix $R(\boldsymbol{\alpha}, \beta)$ admits two-sided inverse whenever its recurrence rule $\boldsymbol{\alpha}$ is an invertible $\mathbf{L}$-set and $\beta \in \mathbf{R}$. Furthermore,*

$$R(\boldsymbol{\alpha}, \beta)^{-1} = R(\tilde{\boldsymbol{\alpha}}, \beta^{-1} \circ (\tilde{\boldsymbol{\alpha}})).$$

The *normalization* of an invertible recursive matrix $R(\boldsymbol{\alpha}, \beta)$ is meant to be the invertible recursive matrix $R(\boldsymbol{\alpha}', \beta)$, where $\boldsymbol{\alpha}'$ is the normalization of the

L-set $\boldsymbol{\alpha}$. An invertible recursive matrix which coincides with its normalization will be called *normalized.*

(3.6) THEOREM. *Let* $\boldsymbol{\alpha} = (\alpha_s)$ *and* $\boldsymbol{\beta} = (\beta_s)$ *be two* S-*sets, with* $\alpha_s, \beta_s \in \mathbf{R}$; *then the equations*

$$\boldsymbol{\alpha}(\mathrm{G})\,M = \mathrm{F}M, \qquad \boldsymbol{\beta}(\mathrm{G})\,TM = \mathrm{F}TM$$

have a nontrivial common solution in the $\mathbb{A}$-*module of all diagonally finite matrices if and only if both* $\boldsymbol{\alpha}$ *and* $\boldsymbol{\beta}$ *are invertible* L-*sets, and* $\boldsymbol{\beta} = \tilde{\boldsymbol{\alpha}}$.

Proof. Let $\boldsymbol{\alpha}$ be an invertible L-set. The following equations are equivalent:

$$\begin{aligned}
\boldsymbol{\alpha}(\mathrm{G})\,M &= \mathrm{F}M,\\
\mathrm{G}M &= \tilde{\boldsymbol{\alpha}}(\mathrm{F})\,M,\\
T\mathrm{F}TM &= \tilde{\boldsymbol{\alpha}}(\mathrm{F})\,M,\\
\mathrm{F}TM &= T\tilde{\boldsymbol{\alpha}}(\mathrm{F})\,TTM,\\
\mathrm{F}TM &= \tilde{\boldsymbol{\alpha}}(\mathrm{G})\,TM.
\end{aligned}$$

Conversely, suppose that there exists a nonzero diagonally finite matrix M such that $\boldsymbol{\alpha}(\mathrm{G})\,M = \mathrm{F}M$ and $\boldsymbol{\beta}(\mathrm{G})\,TM = \mathrm{F}TM$. This implies that both $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are L-sets, and the composition $\boldsymbol{\alpha} \circ \boldsymbol{\beta}$ is defined. Then

$$((\boldsymbol{\alpha} \circ \boldsymbol{\beta})\,\mathrm{F})\,M = (\boldsymbol{\alpha}(\boldsymbol{\beta}(\mathrm{F})))\,M = \boldsymbol{\alpha}(\mathrm{G})\,M = \mathrm{F}M.$$

It follows that $\boldsymbol{\alpha} \circ \boldsymbol{\beta} = \boldsymbol{\tau}$. This completes the proof. ∎

Note that, by the previous theorem, if M is a recursive matrix whose recurrence rule $\boldsymbol{\alpha}$ is an invertible L-set, then TM is also a recursive matrix, with recurrence rule $\tilde{\boldsymbol{\alpha}}$. In particular, if M is any Appell matrix, TM is again an Appell matrix with the same boundary value.

Our next goal is now to determine the boundary value of TM for every invertible recursive matrix M. First of all, we have

(3.7) PROPOSITION. *Let* $\boldsymbol{\alpha}$ *be an invertible* L-*set*; *then*

$$TR(\boldsymbol{\alpha}, P(\boldsymbol{\alpha})) = R(\tilde{\boldsymbol{\alpha}}, \upsilon)$$

where $P(\boldsymbol{\alpha})$ *is the Laurent series defined in Section* 2.

Proof. Follows immediately by Proposition 3.1 and Theorem 2.14. ∎

(3.8) THEOREM. *Let* $\boldsymbol{\alpha}$ *be an invertible* **L**-*set and* β *any Laurent series; then*

$$TR(\boldsymbol{\alpha}, \beta) = R(\tilde{\boldsymbol{\alpha}}, P(\tilde{\boldsymbol{\alpha}})(\beta \circ \tilde{\boldsymbol{\alpha}})).$$

Proof. We have

$$\begin{aligned} TR(\boldsymbol{\alpha}, \beta) &= T(R(\boldsymbol{\alpha}, \upsilon) \times R(\boldsymbol{\tau}, \beta)) = TR(\boldsymbol{\tau}, \beta) \times TR(\boldsymbol{\alpha}, \upsilon) \\ &= R(\boldsymbol{\tau}, \beta) \times R(\tilde{\boldsymbol{\alpha}}, P(\tilde{\boldsymbol{\alpha}})) = R(\tilde{\boldsymbol{\alpha}}, P(\tilde{\boldsymbol{\alpha}})(\beta \circ \tilde{\boldsymbol{\alpha}})). \quad \blacksquare \end{aligned}$$

From Proposition 3.7 we get the following generalized version of the Lagrange–Good formula:

(3.9) THEOREM. *Let* $\boldsymbol{\alpha} = (\alpha_s)$ *be an invertible* **L**-*set and let* $\tilde{\boldsymbol{\alpha}} = (\tilde{\alpha}_s)$ *be its inverse; for every* $s \in S$ *and* $\mathbf{d} \in \mathbf{D}^+$, *we have*

$$\langle \mathbf{d} | \tilde{\alpha}_s \rangle = \langle -\mathbf{e}_s | \boldsymbol{\alpha}^{-\mathbf{d}} P(\boldsymbol{\alpha}) \rangle.$$

In the sequel, we will sometimes need an analog of the notion of recursive matrix for maps $M: \mathbf{D}^+ \times \mathbf{D}^+ \to \mathbb{A}$. In order to avoid confusion, such maps will be called *minors.*

Given any **S**-set $\boldsymbol{\alpha} = (\alpha_s)$ with $\alpha_s \in \mathbf{P}$, and given $\beta \in \mathbf{P}$, $\beta \neq \zeta$, the $(\boldsymbol{\alpha}, \beta)$-*recursive minor* will be the (unique) minor M such that

$$\langle M | \mathbf{d} \rangle = \boldsymbol{\alpha}^d \beta$$

for every $\mathbf{d} \in \mathbf{D}^+$. Such a minor will be denoted by $M(\boldsymbol{\alpha}, \beta)$. $\boldsymbol{\alpha}$ is the *recurrence rule* and β the *boundary value* of $M(\boldsymbol{\alpha}, \beta)$.

The product of two recursive minors is defined whenever the recurrence rule of the second one is a **P**-set. In fact, we have

(3.10) PROPOSITION. *Let* $\boldsymbol{\alpha} = (\alpha_s)$ *be an* **S**-*set, with* $\alpha_s \in \mathbf{P}$, $\boldsymbol{\gamma}$ *a* **P**-set *and* $\beta, \delta \in \mathbf{P}$, $\beta \neq \zeta \neq \delta$. *Then*

$$M(\boldsymbol{\alpha}, \beta) \times M(\boldsymbol{\gamma}, \delta) = M(\boldsymbol{\alpha} \circ \boldsymbol{\gamma}, (\beta \circ \boldsymbol{\gamma})\, \delta).$$

Proof. The same as that of Theorem 3.3. $\blacksquare$

In some cases, if possible, it will be useful to regard a recursive minor as a submatrix of a recursive matrix; precisely, if $M = (m_{\mathbf{f},\mathbf{g}})$ is any matrix, its *Wiener–Hopf truncation* will be the minor

$$WM := (m_{\mathbf{f},\mathbf{g}}) \qquad \text{with} \quad \mathbf{f}, \mathbf{g} \in \mathbf{D}^+.$$

Obviously, a recursive minor can be seen as the Wiener–Hopf truncation of a recursive matrix whenever its recurrence rule consists of series in $\mathbf{P} \cap \mathbf{R}$.

Conversely, the Wiener–Hopf truncation of a recursive matrix is a recursive minor if and only if its recurrence rule consists of series in $\mathbf{P} \cap \mathbf{R}$ and its boundary value is a power series. In particular, the Wiener–Hopf truncation of any permutation matrix is a recursive minor, which will be called a *permutation minor.*

4. Factorial Functions

In the present section we are concerned with a copy $\mathscr{F}$ of the complete topological $\mathbb{A}$-module $\mathbf{S}$ defined in §2. The elements of $\mathscr{F}$ will be called *functions*, and they will be denoted by small script letters. The value taken by the function f at the degree $\mathbf{d}$ will be denoted by $f(\mathbf{d})$ or by $\langle f \mid \mathbf{d} \rangle$. The null function will be denoted by z.

For any given finite subset A of S and for every degree $\mathbf{d} \in \mathbf{D}$, the degree $\chi_A \mathbf{d}$ is defined as follows:

$$\begin{aligned} \chi_A \mathbf{d}(s) &:= \mathbf{d}(s) \qquad \text{if } s \in A, \\ &:= 0 \qquad \text{otherwise.} \end{aligned}$$

We define a class of continuous endomorphisms R_A of $\mathscr{F}$, setting for every finite subset A of S

$$R_A f(\mathbf{d}) = f(\chi_A \mathbf{d})$$

for every $f \in \mathscr{F}$ and $\mathbf{d} \in \mathbf{D}$.

Given a function $f \in \mathscr{F}$, the *canonical approximation* of f will be the sequence $(R_A f)$, where A ranges over the filter of all finite subsets of S. Obviously, $(R_A f)$ converges to f. For every $s \in S$, the sth *difference operator* Δ_s will be the continuous endomorphism of $\mathscr{F}$ such that for every $f \in \mathscr{F}$

$$\Delta_s f(\mathbf{d}) = f(\mathbf{d} + \mathbf{e}_s) - f(\mathbf{d}).$$

For every positive degree $\mathbf{g} = (g_s) \in \mathbf{D}^+$ we set

$$\Delta^{\mathbf{g}} = \prod_{s \in S} \Delta_s^{g_s}.$$

A *pseudo-factorial function* will be a function $f \in \mathscr{F}$ such that for every $s \in S$ there exists $h_s \in \mathbb{N}$ for which

$$\Delta_s^{h_s+1} f = z \qquad \text{and} \qquad \Delta_s^{h_s} f \neq z.$$

The sequence $\mathbf{h} = (h_s)$ will be called the *pseudo-degree* of f. If $\mathbf{h} \in \mathbf{D}^+$, that is, if supp($\mathbf{h}$) is finite, f will be called a *factorial function* and $\mathbf{h}$ will be its *degree*. The null function z will be considered as a factorial function with degree $+\infty$.

A function $f \in \mathscr{E}$ will be said to be a *locally factorial function* whenever for every finite subset A of S $R_A f$ is a factorial function.

The submodules of all locally factorial, pseudo-factorial and factorial functions will be denoted by $\mathscr{H}$, $\mathscr{G}$, and $\mathscr{F}$, respectively. Obviously, $\mathscr{F} \subseteq \mathscr{G} \subseteq \mathscr{H}$, with proper inclusions only if S is infinite.

We have

(4.1) PROPOSITION. *The module $\mathscr{H}$ of locally factorial functions is the topological closure of the module $\mathscr{F}$ of factorial functions.*

Proof. By definition, $\mathscr{H} \subseteq \bar{\mathscr{F}}$. Let now (f_λ) be a sequence in $\mathscr{F}$ with λ belonging to a filter Λ and suppose that (f_λ) converges to $f \in \mathscr{E}$; then, for every finite subset A of S, we have

$$R_A f_\lambda = R_A(\lim_\Lambda f_\lambda) = \lim_\Lambda R_A f_\lambda,$$

where $\lim_\Lambda R_A f_\lambda$ is a factorial function. Hence, f is in $\mathscr{H}$ and $\bar{\mathscr{F}} \subseteq \mathscr{H}$. ∎

We now introduce a family $(b_{\mathbf{d}})$, $\mathbf{d} \in \mathbf{D}^+$, of factorial functions, which will be named *Pascal functions*, setting, for every $\mathbf{d} = (d_s) \in \mathbf{D}^+$ and $\mathbf{h} = (h_s) \in \mathbf{D}$

$$b_{\mathbf{d}}(\mathbf{h}) = \prod_{s \in S} \binom{h_s}{d_s},$$

where the binomial coefficients $\binom{h_s}{d_s}$ are defined recursively as follows:

$$\binom{m}{0} := 1 \qquad \text{for every } m \in \mathbb{Z};$$

$$\binom{0}{n} := 0 \qquad \text{for every } n \in \mathbb{N},\ n \neq 0;$$

$$\binom{m+1}{n+1} := \binom{m}{n} + \binom{m}{n+1} \qquad \text{for every } m \in \mathbb{Z} \text{ and } n \in \mathbb{N},\ n \neq 0.$$

It is straightforward to verify that

$$\begin{aligned} \Delta_s b_{\mathbf{d}} &= b_{\mathbf{d}-\mathbf{e}_s} && \text{if } s \in \operatorname{supp}(\mathbf{d}) \\ &= z && \text{otherwise.} \end{aligned}$$

Hence, $b_{\mathbf{d}}$ has degree exactly $\mathbf{d}$. Furthermore, $b_{\mathbf{d}}(\mathbf{h}) \neq 0$ if and only if, for every $s \in S$, $h_s \geqslant 0$ implies $h_s \geqslant d_s \geqslant 0$.

A power series $\alpha := (a_{\mathbf{d}})$ will be said to be a *pseudo-bounded power series* whenever there exists a pseudo-degree $\mathbf{h}$ such that, if $a_{\mathbf{d}} \neq 0$, then $\mathbf{d} < \mathbf{h}$. A *locally bounded power series* will be a power series $\alpha := (a_{\mathbf{d}})$ such that, for every finite subset A of E, the set

$$\operatorname{supp}_A \alpha := \operatorname{supp} \alpha \cap \{\mathbf{d} \in \mathbf{D}^+;\ \operatorname{supp}(\mathbf{d}) \subseteq A\}$$

is finite.

(4.2) PROPOSITION. *Let $\alpha := (a_{\mathbf{d}})$ be a power series; the sum*

$$\sigma := \sum_{\mathbf{d} \in \mathbf{D}+} a_{\mathbf{d}} \ell_{\mathbf{d}}$$

converges in $\mathscr{X}$ if and only if α is a locally bounded power series.

Proof. Suppose that σ converges in $\mathscr{X}$; for every finite subset A of S, set

$$\mathbf{u}_A := -\sum_{s \in A} \mathbf{e}_s;$$

then, for every $\mathbf{d} \in \mathbf{D}^+$ with $\operatorname{supp}(\mathbf{d}) \subseteq A$, $\ell_{\mathbf{d}}(\mathbf{u}_A) = \pm 1$, which implies that $\operatorname{supp}_A \alpha$ is finite.

Conversely, suppose $\operatorname{supp}_A \alpha$ finite for every finite subset A of S; then, for every $\mathbf{h} \in \mathbf{D}$, the sum

$$\sum_{\mathbf{d} \in \mathbf{D}+} a_{\mathbf{d}} \ell_{\mathbf{d}}(\mathbf{h}) = \sum_{\mathbf{d} \in \operatorname{supp}_A \alpha} a_{\mathbf{d}} \ell_{\mathbf{d}}(\mathbf{h})$$

is finite, and σ converges. ∎

As an example, set

$$\mathbf{F} := \{\mathbf{d} \in \mathbf{D}^+;\ d_s = \operatorname{card} \operatorname{supp}(\mathbf{d}) \text{ for every } s \in \operatorname{supp}(\mathbf{d})\}$$

and

$$\begin{aligned} a_{\mathbf{d}} &:= 1 \qquad \text{if } \ \mathbf{d} \in \mathbf{F}, \\ &:= 0 \qquad \text{otherwise.} \end{aligned}$$

Then, by the preceding result, the sum

$$\sigma := \sum_{\mathbf{d}} a_{\mathbf{d}} \ell_{\mathbf{d}}$$

converges in $\mathscr{X}$. If S is not finite, s is a locally factorial function but not a pseudo-factorial function.

The next theorem shows that $(\ell_{\mathbf{d}})$ is a basis for $\mathscr{F}$.

(4.3) THEOREM (Newton expansion formula). *A function* $f\colon \mathbf{D} \to \mathbb{A}$ *is a factorial function if and only if*

$$f = \sum_{\mathbf{d} \in \mathbf{D}^+} a_{\mathbf{d}} b_{\mathbf{d}},$$

where all but a finite number of coefficients $a_{\mathbf{d}}$ *equal zero.*

Furthermore, for every $\mathbf{d} \in \mathbf{D}^+$,

$$a_{\mathbf{d}} = \langle \Delta^{\mathbf{d}} f \mid \mathbf{0} \rangle,$$

and the degree of f *is* $\mathbf{h} = (h_s)$, *where* $h_s = \max\{d_s\,;\, a_{\mathbf{d}} \neq 0\}$.

Proof. Suppose $f \in \mathscr{F}$ and let $\mathbf{h}$ be the degree of f. We proceed by induction on $k = w(\mathbf{h})$. If $k = 0$ the statement is trivial. Suppose now that the expansion holds whenever $k \leqslant n$, and suppose $w(\mathbf{h}) = n + 1$. Then, for every $s \in \operatorname{supp}(\mathbf{h})$, $\Delta_s f$ is a factorial function whose degree has weight n. By induction hypothesis

$$\Delta_s f = \sum c_{s,\mathbf{d}} b_{\mathbf{d}},$$

where

$$c_{s,\mathbf{d}} = \langle \Delta^{\mathbf{d}+\mathbf{e}_s} f \mid \mathbf{0} \rangle.$$

Set now, for every $\mathbf{d} \in \mathbf{D}^+$

$$a_{\mathbf{d}+\mathbf{e}_s} = c_{s,\mathbf{d}} \qquad \text{and} \qquad a_{\mathbf{0}} = \langle f \mid \mathbf{0} \rangle.$$

We have to prove that, for every $\mathbf{x} \in \mathbf{D}$

$$\langle f \mid \mathbf{x} \rangle = \left\langle \sum a_{\mathbf{d}} b_{\mathbf{d}} \,\middle|\, \mathbf{x} \right\rangle.$$

For $\mathbf{x} = \mathbf{0}$ the identity holds. Suppose the statement true for $\mathbf{x} \in \mathbf{D}$; then, for every $s \in S$,

$$\begin{aligned}
\langle f \mid \mathbf{x} + \mathbf{e}_s \rangle &= \langle f \mid \mathbf{x} \rangle + \langle \Delta_s f \mid \mathbf{x} \rangle \\
&= \left\langle \sum a_{\mathbf{d}} b_{\mathbf{d}} \,\middle|\, \mathbf{x} \right\rangle + \left\langle \sum c_{s,\mathbf{d}} b_{\mathbf{d}} \,\middle|\, \mathbf{x} \right\rangle \\
&= \left\langle \sum a_{\mathbf{d}} b_{\mathbf{d}} \,\middle|\, \mathbf{x} \right\rangle + \left\langle \sum a_{\mathbf{d}} b_{\mathbf{d}-\mathbf{e}_s} \,\middle|\, \mathbf{x} \right\rangle \\
&= \left\langle \sum a_{\mathbf{d}} b_{\mathbf{d}} \,\middle|\, \mathbf{x} + \mathbf{e}_s \right\rangle,
\end{aligned}$$

and analogously

$$\langle f \,|\, \mathbf{x}-\mathbf{e}_s\rangle = \left\langle \sum a_{\mathbf{d}} \ell_{\mathbf{d}} \,|\, \mathbf{x}-\mathbf{e}_s \right\rangle.$$

This completes the proof. ∎

The sequence $(\ell_{\mathbf{d}})$ turns out to be a pseudobasis for $\mathscr{K}$. More precisely, we have

(4.4) THEOREM. *A function $f: \mathbf{D} \to \mathbb{A}$ is a locally factorial function if and only if*

$$f = \sum_{\mathbf{d} \in \mathbf{D}^+} a_{\mathbf{d}} \ell_{\mathbf{d}},$$

where the series $a := (a_{\mathbf{d}})$ is a locally bounded power series, and for every $\mathbf{d} \in \mathbf{D}^+$

$$a_{\mathbf{d}} = \langle \Delta^{\mathbf{d}} f \,|\, \mathbf{0}\rangle.$$

Furthermore, f is a pseudo-factorial function if and only if a is a pseudo-bounded series.

Proof. Follows from Theorem 4.3. ∎

Proposition 4.1 and Theorem 4.4 allow us to restrict our attention only to factorial functions.

5. RECURSIVE SEQUENCES OF FACTORIAL FUNCTIONS

Let $(f_{\mathbf{d}})$, $\mathbf{d} \in \mathbf{D}^+$, be a sequence of functions in $\mathscr{E}$; the *value tableau* of $(f_{\mathbf{d}})$ is the matrix $M := (m_{\mathbf{h},\mathbf{k}})$ defined as follows:

$$\begin{aligned} m_{\mathbf{h},\mathbf{k}} &:= f_{\mathbf{k}}(\mathbf{h}) && \text{if} \quad \mathbf{k} \in \mathbf{D}^+, \\ &:= 0 && \text{otherwise.} \end{aligned}$$

The value tableau of the Pascal functions $(\ell_{\mathbf{d}})$ will be denoted by B. In the sequel, the value tableau of a given sequence $(f_{\mathbf{d}})$ will be sometimes denoted by the same symbol $(f_{\mathbf{d}})$.

A sequence $(f_{\mathbf{d}})$ of functions in $\mathscr{F}$ (or in $\mathscr{S}$) will be called a *factorial sequence* (or a *pseudo-factorial sequence*). If $(f_{\mathbf{d}})$, $\mathbf{d} \in \mathbf{D}^+$, is any factorial sequence, its *canonical minor* will be the minor $C := (c_{\mathbf{h},\mathbf{k}})$ whose entries are $c_{\mathbf{h},\mathbf{k}} := \langle \Delta^{\mathbf{h}} f_{\mathbf{k}} \,|\, \mathbf{0}\rangle$. Plainly, C is the minor of coefficients of $(f_{\mathbf{d}})$ with respect to the basis of the Pascal functions, hence C has finite columns. We remark that the product $B \times C$ is defined and gives the value tableau of the sequence

$$(f_{\mathbf{d}}) = B \times C.$$

A sequence $(f_{\mathbf{d}})$ in $\mathscr{F}$ is said to be a *recursive sequence* if its value tableau is a recursive matrix, that is, if there exists a doubly indexed family $(a_{s,\mathbf{d}})$ of elements in $\mathbb{A}$, with $s \in S$ and $\mathbf{d} \in \mathbf{D}^+$, such that for every $s \in S$ and for every $\mathbf{x} \in \mathbf{D}$, $\mathbf{d} \in \mathbf{D}^+$:

$$f_{\mathbf{d}}(\mathbf{x} + \mathbf{e}_s) = \sum_{\mathbf{g} \in \mathbf{D}} a_{s,\mathbf{g}} f_{\mathbf{d}-\mathbf{g}}(\mathbf{x}).$$

The *recurrence rule* of $(f_{\mathbf{d}})$ will be the set of power series $\{\alpha_s ; s \in S\}$,

$$\alpha_s = \sum_{\mathbf{d} \in \mathbf{D}^+} a_{s,\mathbf{d}} \boldsymbol{\tau}^{\mathbf{d}};$$

the *boundary value* of $(f_{\mathbf{d}})$ will be the power series

$$\beta = \sum_{\mathbf{d} \in \mathbf{D}^+} f_{\mathbf{d}}(\mathbf{0})\, \boldsymbol{\tau}^{\mathbf{d}}.$$

Obviously, a sequence $(f_{\mathbf{d}})$, $\mathbf{d} \in \mathbf{D}^+$, of functions in $\mathscr{F}$ is a recursive sequence if and only if there exist power series (α_s), $s \in S$, with $\langle \mathbf{0} | \alpha_s \rangle \in \mathbb{U}$, its recurrence rule, and β, its boundary value, such that for every $\mathbf{x} \in \mathbf{D}$

$$\sum_{\mathbf{d} \in \mathbf{D}^+} f_{\mathbf{d}}(\mathbf{x})\, \boldsymbol{\tau}^{\mathbf{d}} = \boldsymbol{\alpha}^{\mathbf{x}} \beta.$$

The sequence of Pascal functions is a recursive sequence, whose value tableau is the recursive matrix $R(\boldsymbol{\upsilon} + \boldsymbol{\tau}, \upsilon)$, where $\boldsymbol{\upsilon} + \boldsymbol{\tau} := (\upsilon + \tau_s)$.

In the sequel, the following result will be useful:

(5.1) Proposition. *Let* $\boldsymbol{\alpha} := (\alpha_s)$ *be an* $\mathbf{S}$*-set with* $\deg \alpha_s = \mathbf{0}$ *for every* $s \in S$, *let* $\boldsymbol{\gamma}$ *be a* $\mathbf{P}$*-set, and* β, $\delta \in \mathbf{P}$. *Then*

$$R(\boldsymbol{\alpha}, \beta) \times M(\boldsymbol{\gamma}, \delta) = R(\boldsymbol{\alpha} \circ \boldsymbol{\gamma}, (\beta \circ \boldsymbol{\gamma})\, \delta).$$

Proof. The same as Theorem 3.3. ∎

We have

(5.2) Proposition. *Let* $(f_{\mathbf{d}})$ *be a recursive sequence with recurrence rule* $\boldsymbol{\alpha} := (\alpha_s)$. *If, for some* $\mathbf{h} \in \mathbf{D}^+$, $f_{\mathbf{h}}$ *is a pseudo-factorial function, then* $(f_{\mathbf{d}})$ *is a pseudo-factorial sequence and for every* $s \in S$, $\langle \mathbf{0} | \alpha_s \rangle = 1$. *Conversely, if for every* $s \in S$ $\langle \mathbf{0} | \alpha_s \rangle = 1$, *then* $(f_{\mathbf{d}})$ *is a pseudo-factorial recursive sequence.*

Proof. Setting, for every $s \in S$, $a_s = \langle \mathbf{0} | \alpha_s \rangle$, we have, in any case

(i) $f_{\mathbf{0}}(\mathbf{x}) = f_{\mathbf{0}}(0) \prod_{s \in S} a_s^{\mathbf{x}(s)}$;

(ii) $\Delta_s f_{\mathbf{d}} = (a_s - 1) f_{\mathbf{d}} + \sum_{\mathbf{e} < \mathbf{d}} \langle \mathbf{d} - \mathbf{e} | \alpha_s \rangle f_{\mathbf{e}}$ for every $s \in S$.

Suppose $f_{\mathbf{0}}$ is a pseudo-factorial function; by (ii) we have, for every $s \in S$,

$$\Delta_s f_{\mathbf{0}} = (a_s - 1) f_{\mathbf{0}}$$

which implies $a_s = 1$ for every $s \in S$. By induction, suppose $f_{\mathbf{d}}$ is a pseudo-factorial function for every $\mathbf{d}$ such that $w(\mathbf{d}) \leqslant n$, and let $\mathbf{g} \in \mathbf{D}^+$ such that $w(\mathbf{g}) = n + 1$; by (ii) we get

$$\Delta_s f_{\mathbf{g}} = \sum_{\mathbf{d} < \mathbf{g}} \langle \mathbf{g} - \mathbf{d} | \alpha_s \rangle f_{\mathbf{d}}$$

which implies that $\Delta_s f_{\mathbf{g}}$ is in $\mathscr{S}$, hence $f_{\mathbf{g}}$ itself is in $\mathscr{S}$, and the sequence $(f_{\mathbf{d}})$ is a sequence of pseudo-factorial functions.

Suppose now $f_{\mathbf{h}} \in \mathscr{S}$ for some $\mathbf{h} \in \mathbf{D}^+$, $\mathbf{h} \neq \mathbf{0}$. Set $g_{\mathbf{h}} = f_{\mathbf{h}}$ and, for every $\mathbf{k} \in \mathbf{D}^+$, $\mathbf{k} \leqslant \mathbf{h}$ and for every $s \in \operatorname{supp}(\mathbf{k})$,

$$g_{\mathbf{k} - \mathbf{e}_s} = \Delta_s g_{\mathbf{k}} - (a_s - 1) g_{\mathbf{k}}.$$

Easy computations show that every $g_{\mathbf{k}}$ is a pseudo-factorial function which can be written as a linear combination of functions $f_{\mathbf{d}}$, with $\mathbf{d} \leqslant \mathbf{k}$. This implies that for every $s \in S$ $a_s = 1$ and $f_{\mathbf{0}} \in G$, and we are done.

The second part of the statement is straightforward. ∎

The previous result shows that a recursive sequence is a pseudo-factorial recursive sequence whenever the series of its recurrence rule $\boldsymbol{\alpha} := (\alpha_s)$ are of the form $\alpha_s = \upsilon + \hat{\alpha}_s$, with $w(\hat{\alpha}_s) \geqslant 1$. The set $\hat{\boldsymbol{\alpha}} := (\hat{\alpha}_s)$ will be called the *indicator* of the sequence. Then

(5.3) Proposition. *A pseudo-factorial recursive sequence $(f_{\mathbf{d}})$ is a factorial sequence if and only if its indicator is a* **P**-*set.*

Proof. Let (α_s) be the indicator of the sequence $(f_{\mathbf{d}})$. Suppose (α_s) is a **P**-set. As we remarked above, for every $s \in S$ and $\mathbf{d} \in \mathbf{D}^+$ we have

$$\Delta_s f_{\mathbf{d}} = \sum_{\mathbf{0} < \mathbf{e} \leqslant \mathbf{d}} \langle \mathbf{e} | \alpha_s \rangle f_{\mathbf{d} - \mathbf{e}};$$

for a fixed $\mathbf{d}$, there exists only a finite number of $s \in S$ such that supp α_s contains at least one degree between $\mathbf{0}$ and $\mathbf{d}$; hence, $\Delta_s f_{\mathbf{d}} \neq \varkappa$ for a finite number of $s \in S$.

Conversely, suppose that every $f_{\mathbf{d}}$ is a factorial function; we have to show that, for every $\mathbf{d} \in \mathbf{D}^+$, there is only a finite number of $s \in S$ such that

$\mathbf{d} \in \operatorname{supp} \alpha_s$. We proceed by induction on $h = w(\mathbf{d})$. For $h = 1$ the assertion is true, since

$$\Delta_t f_{\mathbf{e}_s} = \langle \mathbf{e}_s | \alpha_t \rangle f_{\mathbf{0}},$$

that is, $\Delta_t f_{\mathbf{e}_s} \neq \varkappa$ whenever $\langle \mathbf{e}_s | \alpha_t \rangle \neq 0$. Suppose now the statement true for $h \leqslant n$, and choose $\mathbf{d} \in \mathbf{D}^+$, $w(\mathbf{d}) = n + 1$. Then

$$\Delta_s f_{\mathbf{d}} = \sum_{\mathbf{0}<\mathbf{e}<\mathbf{d}} \langle \mathbf{e} | \alpha_s \rangle f_{\mathbf{d}-\mathbf{e}} + \langle \mathbf{d} | \alpha_s \rangle f_{\mathbf{0}} \tag{$*$}$$

and the sets $T = \{s \in S; \Delta_s f_{\mathbf{d}} \neq \varkappa\}$, $Q = \{s \in S;\ \langle \mathbf{e} | \alpha_s \rangle \neq 0 \text{ for some } \mathbf{e} \leqslant \mathbf{d}\}$ are finite. If $s \in S - (T \cup Q)$, $(*)$ implies that $\langle \mathbf{d} | \alpha_s \rangle = 0$, which gives the assertion. ∎

(5.4) PROPOSITION. *A recursive sequence* $(f_{\mathbf{d}})$ *is a factorial recursive sequence if and only if its canonical minor is a recursive minor with a* **P**-*set as recurrence rule. Furthermore, this recurrence rule is precisely the indicator of* $(f_{\mathbf{d}})$ *and the boundary value of the canonical minor is the boundary value of the sequence* $(f_{\mathbf{d}})$.

Proof. Follows immediately by Propositions 5.1 and 5.3. ∎

A factorial recursive sequence which is a basis for $\mathscr{F}$ will be called a *recursive basis*. Then

(5.5) PROPOSITION. *A factorial recursive sequence is a recursive basis if and only if its indicator is an invertible* **P**-*set and its boundary value is in* R.

The coefficients of any factorial function with respect to a given recursive basis can be expressed by means of an "umbral version" of the Newton expansion formula. In order to do this, we remark that, for any given formal power series $\alpha := \sum_{\mathbf{d} \in \mathbf{D}^+} a_{\mathbf{d}} \boldsymbol{\tau}^{\mathbf{d}}$, the formal writing

$$\alpha(\boldsymbol{\Delta}) := \sum_{\mathbf{d} \in \mathbf{D}^+} a_{\mathbf{d}} \boldsymbol{\Delta}^{\mathbf{d}}$$

defines a liner operator on $\mathscr{F}$, since, for every $f \in \mathscr{F}$, $\alpha(\boldsymbol{\Delta}) f$ makes sense and yields a factorial function.

We have

(5.6) THEOREM. *Let* $(f_{\mathbf{d}})$ *be a recursive basis with indicator* $\boldsymbol{\alpha}$ *and boundary value* β. *A function* $g: \mathbf{D} \to \mathbb{A}$ *is a factorial function if and only if*

$$g = \sum_{\mathbf{d} \in \mathbf{D}^+} c_{\mathbf{d}} f_{\mathbf{d}},$$

where

$$c_{\mathbf{d}} = \langle \tilde{\alpha}^{\mathbf{d}}(\beta \circ \tilde{\alpha})^{-1}(\Delta)\, \mathcal{g} | \mathbf{0} \rangle.$$

Proof. Let F be the value tableau of the recursive basis $(f_{\mathbf{d}})$, and let C be its canonical minor. Then we have $B = F \times C^{-1}$, where

$$(a_{\mathbf{h},\mathbf{k}}) = C^{-1} = M(\tilde{\alpha}, (\beta \circ \tilde{\alpha})^{-1}).$$

By the Newton expansion formula, $\mathcal{g}$ is a factorial function if and only if there exists a column vector $E = (e_{\mathbf{d}})$, $\mathbf{d} \in \mathbf{D}^+$, with $e_{\mathbf{d}} = \langle \Delta^{\mathbf{d}} \mathcal{g} | \mathbf{0} \rangle$, such that the column vector $G := (\mathcal{g}(\mathbf{d}))$, $\mathbf{d} \in \mathbf{D}$, can be written as $G = B \times E$. Then, we have $G = F \times C^{-1} \times E$. Setting now $(c_{\mathbf{d}}) := C^{-1} \times E$, we get

$$\begin{aligned} c_{\mathbf{d}} &= \sum_{\mathbf{h}} a_{\mathbf{d},\mathbf{h}} e_{\mathbf{h}} = \sum_{\mathbf{h}} a_{\mathbf{d},\mathbf{h}} \langle \Delta^{\mathbf{h}} \mathcal{g} | \mathbf{0} \rangle \\ &= \left\langle \left(\sum_{\mathbf{h}} a_{\mathbf{d},\mathbf{h}} \Delta^{h} \right) \mathcal{g} \,\middle|\, \mathbf{0} \right\rangle = \langle \tilde{\alpha}^{\mathbf{d}}(\beta \circ \tilde{\alpha})^{-1}(\Delta)\, \mathcal{g} | \mathbf{0} \rangle. \quad \blacksquare \end{aligned}$$

We are now interested in characterizing linear operators $L : \mathcal{F} \to \mathcal{F}$ which preserve recursivity. A linear operator L will be said a *recursive operator* whenever there exists a recursive basis $(f_{\mathbf{d}})$ such that its image $(Lf_{\mathbf{d}})$ is a factorial recursive sequence.

First of all, given any linear operator $L : \mathcal{F} \to \mathcal{F}$, its *canonical minor* is meant as the canonical minor of the sequence $(L\ell_{\mathbf{d}})$; in the sequel, the same symbol will denote both the operator and its canonical minor.

If $(f_{\mathbf{d}})$ is a basis of $\mathcal{F}$, with canonical minor C, we get

$$(Lf_{\mathbf{d}}) = (L(B \times C)) = B \times L \times C = (f_{\mathbf{d}}) \times C^{-1} \times L \times C.$$

(5.7) PROPOSITION. *Let $L : \mathcal{F} \to \mathcal{F}$ be a linear operator. The following are equivalent*:

(i) *L is a recursive operator*;

(ii) *the canonical minor of L is a recursive minor*;

(iii) *L maps every recursive basis into a recursive sequence.*

Proof. Let L be a recursive operator which maps the recursive basis $(f_{\mathbf{d}}) = B \times C$ into the recursive sequence $(Lf_{\mathbf{d}})$. Then, the minor $C^{-1} \times L \times C$ is a recursive minor; thus, L is also a recursive minor.

If the canonical minor of L is a recursive minor, for every recursive basis $(f_{\mathbf{d}})$ with canonical minor C we have

$$(Lf_{\mathbf{d}}) = (f_{\mathbf{d}}) \times C^{-1} \times L \times C$$

which implies that $(Lf_{\mathbf{d}})$ is a recursive sequence. $\blacksquare$

Given a recursive operator L, with canonical minor $M(\boldsymbol{\alpha}, \beta)$, $\boldsymbol{\alpha}$ will be called the *recurrence rule* of L, and the *indicator* of L will be the series $\operatorname{ind}(L) := \beta$.

The preceding result implies that the semigroup of all recursive operators on $\mathscr{F}$ is isomorphic to the semigroup of all recursive minors with a $\mathbf{P}$-set as recurrence rule.

An invertible recursive operator will be called an *umbral operator*. For a given bijection $\sigma: S \to S$, the *permutation operator* P_σ will be the umbral operator whose canonical minor is $M(\boldsymbol{\tau}^\sigma, \upsilon)$.

A linear operator $T: \mathscr{F} \to \mathscr{F}$ satisfying the following conditions:

$$T\Delta_s = \Delta_s T$$

for every $s \in S$, will be called a *shift-invariant operator*. We have

(5.8) PROPOSITION. *Let $T: \mathscr{F} \to \mathscr{F}$ be a linear operator. The following are equivalent*:

(i) *T is shift-invariant*;

(ii) $T = \sum_{\mathbf{d}} \langle T\ell_{\mathbf{d}} | \mathbf{0} \rangle \, \boldsymbol{\Delta}^{\mathbf{d}}$;

(iii) *the canonical minor of T is an Appell minor.*

Proof. We have only to show that (iii) implies (ii), since the other implications are matter of straightforward computations.

Suppose that the canonical minor of T is $(t_{\mathbf{h},\mathbf{k}}) = M(\boldsymbol{\tau}, \alpha)$; then, for every $\mathbf{d} \in \mathbf{D}^+$ we have

$$T\ell_{\mathbf{d}} = \sum_{\mathbf{h}} t_{\mathbf{h},\mathbf{d}} \ell_{\mathbf{h}} = \sum_{\mathbf{h}} t_{\mathbf{0},\mathbf{d}-\mathbf{h}} \ell_{\mathbf{h}} = \sum_{\mathbf{h}} \langle T\ell_{\mathbf{d}-\mathbf{h}} | \mathbf{0} \rangle \, \boldsymbol{\Delta}^{\mathbf{d}-\mathbf{h}} \ell_{\mathbf{d}}$$

which implies (ii). ∎

An immediate consequence of the previous result is that a shift-invariant operator maps each submodule $\mathscr{F}_{\mathbf{d}} := \{ f \in \mathscr{F};\ \deg f \subseteq \mathbf{d} \}$ of $\mathscr{F}$ into itself.

The set Σ of all shift-invariant operators, under the usual sum and composition of functions, turns out to be an $\mathbb{A}$-algebra which is isomorphic, via the map $T \to \operatorname{ind}(T)$, to the algebra $\mathbf{P}$ of formal power series. The algebra Σ can be endowed with a topology by means of the following convergence criterion: a sequence $(T_{\mathbf{d}})$, $T_{\mathbf{d}} \in \Sigma$, $\mathbf{d} \in \mathbf{D}^+$, *converges* to $T \in \Sigma$ whenever, for every $f \in \mathscr{F}$ there exists $\mathbf{d}_f \in \mathbf{D}^+$ such that, for every $\mathbf{d} \geqslant \mathbf{d}_f$: $T_{\mathbf{d}} f = Tf$.

(5.9) PROPOSITION. *The map $T \to \operatorname{ind}(T)$ is a continuous isomorphism between the topological algebras Σ and* $\mathbf{P}$.

Proof. Specializing to the basis of Pascal functions the convergence criterion defined in Σ it is easily seen that a sequence (T_d) in Σ converges to $T \in \Sigma$ if and only if the sequence $(\mathrm{ind}(T_d))$ converges in $\mathbf{P}$ to $\mathrm{ind}(T)$. ∎

The *indicator* of a set $\mathbf{T} := (T_s)$, $T_s \in \Sigma$, $s \in S$, will be the set of power series $\mathrm{ind}(\mathbf{T}) := (\mathrm{ind}(T_s))$. The *indicator* of a linear map $\Phi : \Sigma \to \Sigma$ is $\mathrm{ind}(\Phi) := \mathrm{ind}(\Phi(\Delta))$.

(5.10) PROPOSITION. *A linear map $\Phi : \Sigma \to \Sigma$ is a continuous endomorphism of Σ if and only if* $\mathrm{ind}(\Phi)$ *is a* $\mathbf{P}$*-set. Moreover, for every $T \in \Sigma$,*

$$\mathrm{ind}(\Phi(T)) = (\mathrm{ind}(T)) \circ \mathrm{ind}(\Phi).$$

(5.11) PROPOSITION. *A map $\Phi : \Sigma \to \Sigma$ is the continuous automorphism whose indicator is the invertible* $\mathbf{P}$*-set* $\boldsymbol{\alpha}$ *if and only if for every $T \in \Sigma$*

$$\Phi(T) = U^{-1}TU,$$

where U is any umbral operator whose recurrence rule is $\boldsymbol{\alpha}$.

Proof. Let $M(\boldsymbol{\alpha}, \beta)$ be the canonical minor of U. The map $T \to U^{-1}TU$ is a linear map Ψ, whose indicator is given by the boundary values of the minors

$$M(\tilde{\boldsymbol{\alpha}}, \beta^{-1} \circ \tilde{\boldsymbol{\alpha}}) \times M(\boldsymbol{\tau}, \tau_s) \times M(\boldsymbol{\alpha}, \beta) = M(\boldsymbol{\tau}, \alpha_s),$$

hence, $\mathrm{ind}(\Psi) = \boldsymbol{\alpha}$ and $\Psi = \Phi$. ∎

Generalizing the classical notion of delta operator, we can define a *delta set* as the image of the set Δ under a continuous automorphism Φ of Σ, that is, a set $\mathbf{T} := (T_s)$ of shift-invariant operators such that $\mathrm{ind}(\mathbf{T})$ $(= \mathrm{ind}(\Phi))$ is an invertible $\mathbf{P}$-set.

Following along the lines of [4,23], we say that a given sequence $(q_{\mathbf{d}})$, $\mathbf{d} \in \mathbf{D}^+$, of factorial functions, and a set $\mathbf{Q} = (Q_s)$ of linear operators are *associated* whenever, for every $\mathbf{d} \in \mathbf{D}^+$ and for every $s \in S$

$$\begin{aligned} Q_s q_{\mathbf{d}} &= q_{\mathbf{d}-\mathbf{e}_s} && \text{if } s \in \mathrm{supp}(\mathbf{d}), \\ &= \varkappa && \text{otherwise.} \end{aligned}$$

The sequence $(p_{\mathbf{d}})$ of Pascal functions and the set Δ are associated.

(5.12) PROPOSITION. *Let $(q_{\mathbf{d}})$ be a recursive basis of $\mathscr{F}$ with indicator* $\boldsymbol{\alpha}$

and let U be the umbral operator such that $U\mathscr{b}_{\mathbf{d}} = \mathscr{q}_{\mathbf{d}}$ for every $\mathbf{d} \in \mathbf{D}^+$. Then, the set

$$\mathbf{Q} := U\mathbf{\Delta}U^{-1} = (U\Delta_s U^{-1})$$

is a delta set and it is the unique set of linear operators associated to $(\mathscr{q}_{\mathbf{d}})$. Moreover, $\operatorname{ind}(\mathbf{Q}) = \tilde{\boldsymbol{\alpha}}$.

Proof. Being $(\mathscr{q}_{\mathbf{d}})$ a basis for $\mathscr{F}$, for every $s \in S$ there exists exactly one linear operator Q_s such that $Q_s \mathscr{q}_{\mathbf{d}} = \mathscr{q}_{\mathbf{d}-\mathbf{e}_s}$ if $s \in \operatorname{supp}(\mathbf{d})$, and $Q_s \mathscr{q}_{\mathbf{d}} = \varkappa$ otherwise; hence, the set $\mathbf{Q}$ is uniquely determined.

On the other hand, for every $\mathbf{d} \in \mathbf{D}^+$ and for every $s \in \operatorname{supp}(\mathbf{d})$,

$$(U\Delta_s U^{-1})\mathscr{q}_{\mathbf{d}} = (U\Delta_s)\,\mathscr{b}_{\mathbf{d}} = U\mathscr{b}_{\mathbf{d}-\mathbf{e}_s} = \mathscr{q}_{\mathbf{d}-\mathbf{e}_s}.$$

By the preceding result, $\mathbf{Q}$ is a delta set and

$$\operatorname{ind}(\mathbf{Q}) = \tilde{\boldsymbol{\alpha}}. \quad \blacksquare$$

Conversely,

(5.13) Proposition. *Let $\mathbf{Q} := (Q_s)$ be a delta set with indicator $\boldsymbol{\alpha}$. A sequence $(\mathscr{q}_{\mathbf{d}})$ of factorial functions is associated to $\mathbf{Q}$ if and only if it is a recursive sequence with indicator $\tilde{\boldsymbol{\alpha}}$. Moreover, $(\mathscr{q}_{\mathbf{d}})$ is a recursive basis if and only if its boundary value is in* $\mathbf{R}$.

Proof. By construction, we recognize that $(\mathscr{q}_{\mathbf{d}})$ is uniquely determined if the evaluations $\langle \mathscr{q}_{\mathbf{d}} | \mathbf{0} \rangle$ are given. By Proposition 3.10, the canonical minor of $(\mathscr{q}_{\mathbf{d}})$ is $M(\tilde{\boldsymbol{\alpha}}, \beta)$, where $\beta := \Sigma \langle \mathscr{q}_{\mathbf{d}} | \mathbf{0} \rangle\, \boldsymbol{\tau}^{\mathbf{d}}$ and $\tilde{\boldsymbol{\alpha}}$ is the inverse of $\operatorname{ind}(\mathbf{Q})$. $\blacksquare$

(5.14) Corollary. *Let $(\mathscr{q}_{\mathbf{d}})$ be a recursive basis of $\mathscr{F}$: then, for every $\mathbf{m}, \mathbf{n} \in \mathbf{D}^+$, $\mathbf{m} \leqslant \mathbf{n}$ implies* $\deg(\mathscr{q}_{\mathbf{m}}) \leqslant \deg(\mathscr{q}_{\mathbf{n}})$.

Proof. Let $\mathbf{Q}$ be the delta set associated to $(\mathscr{q}_{\mathbf{d}})$ and recall that shift-invariant operators do not increase the degrees; then

$$\deg(\mathscr{q}_{\mathbf{m}-\mathbf{e}_s}) = \deg(Q_s \mathscr{q}_{\mathbf{m}}) \leqslant \deg(\mathscr{q}_{\mathbf{m}})$$

for every $\mathbf{m} \in \mathbf{D}^+$ and $s \in \operatorname{supp}(\mathbf{m})$. $\blacksquare$

Finally, we give an explicit expansion formula for shift-invariant operators in terms of powers of a given delta set

(5.15) Proposition. *Let $\mathbf{Q} := (Q_s)$ be a delta set and $(\mathscr{q}_{\mathbf{d}})$ the homogeneous recursive basis associated to it. For every $T \in \Sigma$ we have*

$$T = \sum_{\mathbf{d}} \langle T\mathscr{q}_{\mathbf{d}} | \mathbf{0} \rangle\, \mathbf{Q}^{\mathbf{d}}.$$

Proof. Let α be the indicator of $\mathbf{Q}$; for every $T \in \Sigma$, with $\beta := \operatorname{ind}(T)$, we have

$$(\mathrm{T}q_{\mathbf{d}}) = B \times M(\boldsymbol{\tau}, \beta) \times M(\tilde{\boldsymbol{\alpha}}, \upsilon),$$

and, hence

$$\sum_{\mathbf{d}} \langle Tq_{\mathbf{d}} | \mathbf{0} \rangle \, \boldsymbol{\tau}^{\mathbf{d}} = \beta \circ \tilde{\boldsymbol{\alpha}}$$

which implies

$$\sum_{\mathbf{d}} \langle Tq_{\mathbf{d}} | \mathbf{0} \rangle \mathbf{Q}^{\mathbf{d}} = \beta \circ \tilde{\boldsymbol{\alpha}} \circ \boldsymbol{\alpha}(\boldsymbol{\Delta}) = \beta(\boldsymbol{\Delta}) = T. \quad \blacksquare$$

6. Coherence

In this section we introduce the notion of coherence, which allows us to refine the results of the previous section.

First of all, we say that a *multifiltration* $(\mathscr{H}_{\mathbf{d}})$, $\mathbf{d} \in \mathbf{D}^+$, of $\mathscr{F}$ is a sequence of $\mathbb{A}$-submodules of $\mathscr{F}$ such that $\mathbf{h} \leqslant \mathbf{k}$ implies $\mathscr{H}_{\mathbf{h}} \subseteq \mathscr{H}_{\mathbf{k}}$ and $\bigcup_{\mathbf{d} \in \mathbf{D}^+} \mathscr{H}_{\mathbf{d}} = \mathscr{F}$, and a *grading* $(\mathscr{G}_{\mathbf{d}})$, $\mathbf{d} \in \mathbf{D}^+$, of $\mathscr{F}$ is a sequence of $\mathbb{A}$-submodules of $\mathscr{F}$ such that $\mathscr{F} = \bigoplus_{\mathbf{d} \in \mathbf{D}^+} \mathscr{G}_{\mathbf{d}}$. The multifiltration $(\mathscr{H}_{\mathbf{d}})$ is said to be *homogeneous* with respect to the grading $(\mathscr{G}_{\mathbf{d}})$ if, for every $\mathbf{d} \in \mathbf{D}^+$,

$$\mathscr{H}_{\mathbf{d}} = \bigoplus_{\mathbf{t}} (\mathscr{H}_{\mathbf{d}} \cap \mathscr{G}_{\mathbf{t}}).$$

The *Pascal multifiltration* $(\mathscr{F}_{\mathbf{d}})$ of $\mathscr{F}$ is defined by setting, for every $\mathbf{d} \in \mathbf{D}^+$,

$$\mathscr{F}_{\mathbf{d}} := \bigcap_{s \in S} \ker \Delta^{d_s + 1}.$$

A grading $(\mathscr{G}_{\mathbf{d}})$ on F will be called *coherent* whenever the Pascal multifiltration is homogeneous with respect to $(\mathscr{G}_{\mathbf{d}})$.

A basis $(f_{\mathbf{d}})$ of $\mathscr{F}$ will be called a *coherent basis* whenever the grading $([f_{\mathbf{d}}])$ is coherent (here, $[f]$ denotes the $\mathbb{A}$-submodule of $\mathscr{F}$ spanned by f). The Newton expansion formula (Theorem 4.3) ensures that the basis $(\ell_{\mathbf{d}})$ of Pascal functions is a coherent basis.

Coherent bases can be characterized as follows:

(6.1) Proposition. *Let $(f_{\mathbf{d}})$ be a basis for $\mathscr{F}$ and let $f_{\mathbf{d}} = \sum_{\mathbf{t}} c_{\mathbf{d},\mathbf{t}} \ell_{\mathbf{t}}$. Then, $(f_{\mathbf{d}})$ is a coherent basis if and only if the map*

$$\lambda : \mathbf{D}^+ \to \mathbf{D}^+, \qquad \lambda(\mathbf{d}) := \deg(f_{\mathbf{d}})$$

is a bijection and, moreover, for every $\mathbf{d} \in \mathbf{D}^+$, $c_{\mathbf{d}, \lambda(\mathbf{d})} \in \mathbb{U}$.

Proof. For every $\mathbf{h} \in \mathbf{D}^+$, set

$$A_{\mathbf{h}} := \{\mathbf{t} \in \mathbf{D}^+;\ c_{\mathbf{h},\mathbf{t}} \neq 0\}.$$

Then, for every $\mathbf{d}, \mathbf{h} \in \mathbf{D}^+$, $f_{\mathbf{h}} \in \mathscr{F}_{\mathbf{d}}$ if and only if $A_{\mathbf{h}} \subseteq [\mathbf{0}, \mathbf{d}]$.

Given $\mathbf{d} \in \mathbf{D}^+$, the assumption

$$\mathscr{F}_{\mathbf{d}} = \oplus(\mathscr{F}_{\mathbf{d}} \cap [f_{\mathbf{h}}])$$

implies that

$$[\mathbf{0}, \mathbf{d}] = \bigcup_{A_{\mathbf{h}} \subseteq [\mathbf{0},\mathbf{d}]} A_{\mathbf{h}}.$$

Hence, there exists $\mathbf{t} \in \mathbf{D}^+$ such that $\mathbf{d} \in A_{\mathbf{t}} \subseteq [\mathbf{0}, \mathbf{d}]$, that is, $\deg(f_{\mathbf{t}}) = \mathbf{d}$ and $c_{\mathbf{t},\mathbf{d}} \neq 0$.

Suppose now that $(f_{\mathbf{d}})$ is a coherent basis: then, the preceding argument shows that the map λ is a surjection; on the other hand, being $(f_{\mathbf{d}})$ a basis, λ must be a bijection and $c_{\mathbf{d},\lambda(\mathbf{d})} \in \mathbb{U}$ for every $\mathbf{d} \in \mathbf{D}^+$.

The proof of the converse is straightforward. ∎

The next result characterizes coherent recursive bases in terms of their indicator:

(6.2) THEOREM. *A factorial recursive sequence $(f_{\mathbf{d}})$ is a coherent basis of $\mathscr{F}$ if and only if its indicator $\boldsymbol{\alpha} := (\alpha_s)$ is an invertible* $\mathbf{L}$*-set, and its boundary value β is in* $\mathbf{R}$.

Proof. Suppose that $\boldsymbol{\alpha}$ is an invertible $\mathbf{L}$-set and $\langle \mathbf{0} | \beta \rangle \in \mathbb{U}$. Without loss of generality, we can assume that $\boldsymbol{\alpha}$ is normalized. Under these hypotheses, the canonical minor $C := (c_{\mathbf{h},\mathbf{k}})$ of $(f_{\mathbf{d}})$ is an invertible upper triangular minor, with $c_{\mathbf{d},\mathbf{d}} \in \mathbb{U}$; this shows that $(f_{\mathbf{d}})$ is a coherent basis.

Conversely, suppose that the recursive sequence $(f_{\mathbf{d}})$ is a coherent basis and recall that, because of recursivity, $\mathbf{h} \leqslant \mathbf{k}$ implies $\deg(f_{\mathbf{h}}) \leqslant \deg(f_{\mathbf{k}})$. This condition, together with coherence, implies that the map

$$\lambda : \mathbf{d} \to \deg(f_{\mathbf{d}})$$

is a lattice isomorphism. Hence, without loss of generality, we can suppose that λ is the identity map. Let $\mathbf{F} := (F_s)$ be the delta set associated to $(f_{\mathbf{d}})$; then, for every $s \in S$, $\mathbf{d} \in \mathbf{D}^+$,

$$\begin{aligned} \deg(F_s f_{\mathbf{d}}) &= \mathbf{d} - \mathbf{e}_s \qquad && \text{if } s \in \operatorname{supp}(\mathbf{d}) \\ &= +\infty && \text{otherwise.} \end{aligned}$$

Since $F_s = \tilde{\alpha}_s(\boldsymbol{\Delta})$ (with $(\tilde{\alpha}_s)$ the compositional inverse of $\boldsymbol{\alpha}$), this implies that, if $\langle \mathbf{d} | \tilde{\alpha}_s \rangle \neq 0$, then $s \in \operatorname{supp}(\mathbf{d})$, that is, $(\tilde{\alpha}_s)$ is an invertible $\mathbb{L}$-set, and the same holds for the set $\boldsymbol{\alpha}$. ∎

A factorial recursive sequence which is a coherent basis for $\mathscr{F}$ will be said to be a *σ-coherent recursive basis*, where σ is the spire of its indicator. *Normalizations* and *normalized coherent bases* are defined in the usual way (see Section 2).

By using coherent recursive bases, the expansion formula given by Theorem 5.6 can be easily specialized as follows:

(6.2) THEOREM. *Let $(f_{\mathbf{d}})$ be the σ-coherent recursive basis with indicator $\boldsymbol{\alpha}$ and boundary value β. A function $g : \mathbf{D} \to \mathbb{A}$ is a factorial function of degree $\mathbf{t}$ if and only if*

$$g = \sum_{\mathbf{d}} c_{\mathbf{d}} f_{\mathbf{d}},$$

where

$$c_{\mathbf{d}} := \langle \tilde{\boldsymbol{\alpha}}^{\mathbf{d}} (\beta \circ \tilde{\boldsymbol{\alpha}})^{-1} (\boldsymbol{\Delta})\, g | \mathbf{0} \rangle$$

and the sum ranges over all degrees $\mathbf{d}$ such that $\mathbf{0} \leqslant \sigma(\mathbf{d}) \leqslant \mathbf{t}$. ∎

A *σ-coherent umbral operator* will be an umbral operator U which maps the basis $(\ell_{\mathbf{d}})$ of Pascal functions into a σ-coherent recursive basis. In particular, the permutation operator P_σ (see Section 5) is a σ-coherent operator. *Normalizations* and *normalized umbral operators* are defined in the obvious way.

From Proposition 5.7 we have.

(6.3) PROPOSITION. *Let $L : \mathscr{F} \to \mathscr{F}$ be a linear operator and let σ be a bijection on S. The following are equivalent*:

(i) *L is a σ-coherent umbral operator*;

(ii) *the canonical minor of L is a recursive minor whose recurrence rule is an invertible $\mathbb{L}$-set $\boldsymbol{\alpha}$, and σ is the spire of $\boldsymbol{\alpha}$*;

(iii) *L maps every ρ-coherent recursive basis into a σρ-coherent recursive basis.* ∎

Thus, invertible shift-invariant operators are normalized umbral operators.

A continuous automorphism Φ of the algebra Σ of all shift-invariant operators will be said to be a *σ-coherent automorphism* whenever $\operatorname{ind}(\Phi)$ is an (invertible) $\mathbb{L}$-set whose spire is σ. *Normalizations* and *normalized coherent automorphisms* are defined in the usual way.

Obviously, we have (see Proposition 5.11)

(6.4) PROPOSITION. *A map $\Phi : \Sigma \to \Sigma$ is the σ-coherent automorphism whose indicator is the invertible* **L**-*set* $\boldsymbol{\alpha}$ *if and only if, for every $T \in \Sigma$,*

$$\Phi(T) = U^{-1}TU,$$

where U is any umbral operator whose recurrence rule is $\boldsymbol{\alpha}$.

For any continuous automorphism Φ of Σ, we define a multifiltration $(\mathscr{F}^{\Phi}_{\mathbf{d}})$ by setting

$$\mathscr{F}^{\Phi}_{\mathbf{d}} := \bigcap_{s \in S} \ker(\Phi(\Delta_s))^{d_s+1}.$$

(6.5) THEOREM. *A continuous automorphism Φ of Σ is a σ-coherent automorphism if and only if the grading $([\ell_{\mathbf{d}}])$ of Pascal function is homogeneous with respect to the multifiltration $(\mathscr{F}^{\Phi}_{\mathbf{d}})$ and for every $s \in S$,*

$$\mathscr{F}^{\Phi}_{\mathbf{e}_s} = \mathscr{F}_{\mathbf{e}_{\sigma(s)}}.$$

Proof. Let $\boldsymbol{\alpha} := (\alpha_s) = \operatorname{ind}(\Phi)$ and set $\mathbf{Q} := (Q_s) = (\Phi(\Delta_s)) = (\alpha_s(\boldsymbol{\Delta}))$. Suppose Φ is σ-coherent; then, for every $s \in S$, $Q_s = \Delta_{\sigma(s)} \hat{Q}_s$, where $\hat{Q}_s$ is an invertible shift-invariant operator. This implies that for every Pascal function $\ell_{\mathbf{d}}$ we have

$$\begin{aligned} \deg Q_s(\ell_{\mathbf{d}}) &= \mathbf{d} - \mathbf{e}_{\sigma(s)} && \text{if} \quad s \in \operatorname{supp}(\mathbf{d}), \\ &= +\infty && \text{otherwise.} \end{aligned}$$

Hence, for every $\mathbf{d} \in \mathbf{D}^+$, $\mathscr{F}^{\Phi}_{\mathbf{d}} = \mathscr{F}_{\sigma(\mathbf{d})}$, which ensures that the grading of Pascal functions is homogeneous with respect to the multifiltration $(\mathscr{F}^{\Phi}_{\mathbf{d}})$.

Conversely, suppose that the Pascal grading $([\ell_{\mathbf{d}}])$ is homogeneous with respect to the multifiltration $(\mathscr{F}^{\Phi}_{\mathbf{d}})$ and for every $s \in S : \mathscr{F}^{\Phi}_{\mathbf{e}_s} = \mathscr{F}_{\mathbf{e}_{\sigma(s)}}$. Let U be an umbral operator whose recurrence rule is $\boldsymbol{\alpha}$. By Proposition 5.11 we have, for every $T \in \Sigma$, $\Phi(T) = U^{-1}TU$; then,

$$\ker(\Phi(\Delta_s))^m = \ker(U^{-1}\Delta_s^m U) = \ker(\Delta_s^m U) = U^{-1}(\ker \Delta_s^m).$$

This implies, for every $\mathbf{d} \in \mathbf{D}^+$, $\mathscr{F}^{\Phi}_{\mathbf{d}} = U^{-1}(\mathscr{F}_{\mathbf{d}})$. Thus, the grading $([U\ell_{\mathbf{d}}])$ is homogeneous with respect to the Pascal multifiltration $(\mathscr{F}_{\mathbf{d}}) = (U(\mathscr{F}^{\Phi}_{\mathbf{d}}))$. Then, $(U\ell_{\mathbf{d}})$ is a σ-coherent basis, that is, $\boldsymbol{\alpha}$ is an invertible **L**-set whose spire is σ, and Φ is a σ-coherent automorphism of Σ. ∎

By the preceding theorem, we get immediately.

(6.6) COROLLARY. *Let Φ be a continuous automorphism of Σ and $(f_{\mathbf{d}})$ a recursive basis of $\mathscr{F}$. If the grading $([f_{\mathbf{d}}])$ is homogeneous with respect to the multifiltration $(\mathscr{F}^{\Phi}_{\mathbf{d}})$, then Φ is a σ-coherent automorphism (for some* *bijection σ on S) if and only if $(f_{\mathbf{d}})$ is a ρ-coherent basis for some bijection ρ on S.*

Conversely, if there exist two bijections ρ and σ on S such that $(f_{\mathbf{d}})$ is a ρ-coherent basis and Φ is a σ-coherent automorphism, then the grading $([f_{\mathbf{d}}])$ is homogeneous with respect to the multifiltration $(\mathscr{F}_{\mathbf{d}}^{\Phi})$.

A *σ-coherent delta set* (or *σ-delta set* for short) will be the image of the set $\boldsymbol{\Delta}$ under a σ-coherent automorphism Φ of Σ, that is, a set $\mathbf{T} := (T_s)$ of shift-invariant operators such that $\operatorname{ind}(\mathbf{T})$ $(=\operatorname{ind}(\Phi))$ is an invertible $\mathbf{L}$-set whose spire is σ. *Normalizations* and *normalized delta sets* are defined in the usual way.

From Propositions 5.12 and 5.13 we get

(6.7) PROPOSITION. *Let $(f_{\mathbf{d}})$ be a σ-coherent recursive basis of $\mathscr{F}$ with indicator $\boldsymbol{\alpha}$ and let $U: \mathscr{F} \to \mathscr{F}$ be the σ-coherent umbral operator such that $U\ell_{\mathbf{d}} = f_{\mathbf{d}}$ for every $\mathbf{d} \in \mathbf{D}^+$. Then, the set*

$$\mathbf{T} := U\boldsymbol{\Delta}U^{-1} = (U\Delta_s U^{-1})$$

is the unique set of linear operators associated to $(f_{\mathbf{d}})$. Moreover, $\operatorname{ind}(\mathbf{T}) = \tilde{\boldsymbol{\alpha}}$ *and $\mathbf{T}$ is a σ^{-1}-delta set.*

(6.8) PROPOSITION. *Let $\mathbf{T} := (T_s)$ be a σ-delta set and let $\boldsymbol{\alpha}$ be its indicator. A sequence $(f_{\mathbf{d}})$ of factorial functions is associated to $\mathbf{T}$ if and only if it is a recursive sequence with indicator $\tilde{\boldsymbol{\alpha}}$. Moreover, in this case, $(f_{\mathbf{d}})$ is a σ^{-1}-coherent recursive basis whenever its boundary value is in* $\mathbf{R}$.

7. CLOSED FORMS

The purpose of this section is to study those sequences $(f_{\mathbf{d}})$ in $\mathscr{F}$ which are recursive bases and satisfy an analog of the classical transfer formula, that is, such that

$$f_{\mathbf{d}} = F_{\mathbf{d}}\ell_{\sigma(\mathbf{d})} = F_{\mathbf{d}}P_{\sigma}\ell_{\mathbf{d}}, \qquad (*)$$

where σ is a permutation on S, and $(F_{\mathbf{d}})$ is a family of shift-invariant operators.

(7.1) PROPOSITION. *A basis $(f_{\mathbf{d}})$ satisfying a set of identities $(*)$ is a coherent basis, and vice versa.*

Proof. First of all we recall that every shift-invariant operator F does not increase the degrees, that is, for every factorial function g,

$$\deg(Fg) \leqslant \deg(g).$$

Then, if $(p_{\mathbf{d}})$ is a basis satisfying $(*)$, for every $\mathbf{d} \in \mathbf{D}^+$,

$$\deg(p_{\mathbf{d}}) = \deg(F_{\mathbf{d}} P_\sigma b_{\mathbf{d}}) \leqslant \sigma(\mathbf{d}).$$

On the other hand, being $(p_{\mathbf{d}})$ a basis, we have $\deg(p_{\mathbf{d}}) = \sigma(\mathbf{d})$, and the basis $(p_{\mathbf{d}})$ is coherent.

Conversely, let $(p_{\mathbf{d}})$ be a coherent basis, with $\deg(p_{\mathbf{d}}) = \sigma(\mathbf{d})$, and let

$$p_{\mathbf{d}} = \sum_{\mathbf{0} \leqslant \mathbf{t} \leqslant \sigma(\mathbf{d})} c_{\mathbf{d},\mathbf{t}} b_{\mathbf{t}}.$$

Then, $(p_{\mathbf{d}})$ satisfies $(*)$ with

$$F_{\mathbf{d}} = \sum_{\mathbf{0} \leqslant \mathbf{g} \leqslant \sigma(\mathbf{d})} c_{\mathbf{d},\sigma(\mathbf{d})-\mathbf{g}} \Delta^{\mathbf{g}}. \quad \blacksquare$$

By the preceding result, recursive bases satisfying $(*)$ are precisely coherent recursive bases. For such bases, the collection of shift-invariant operators $F_{\mathbf{d}}$ can be recursively computed:

(7.2) THEOREM (Transfer formula). *Let $(p_{\mathbf{d}})$ be a normalized coherent basis with indicator $\boldsymbol{\alpha}$ and boundary value β. Let $\mathbf{Q} := (Q_s)$ be the normalized delta set associated to $(p_{\mathbf{d}})$ and let $\mathbf{T} := (T_s)$ be such that $\Delta_s T_s = Q_s$. Set*

$$B := (\beta \circ \tilde{\boldsymbol{\alpha}})(\boldsymbol{\Delta}) \qquad \text{and} \qquad L := P(\tilde{\boldsymbol{\alpha}})(\boldsymbol{\Delta}).$$

Then the basis $(p_{\mathbf{d}})$ satisfies the Transfer formula

$$p_{\mathbf{d}} = BL\mathbf{T}^{-\mathbf{d}} b_{\mathbf{d}}.$$

Proof. The canonical minor of the basis $(p_{\mathbf{d}})$ is $M(\boldsymbol{\alpha}, \beta)$. Since $\boldsymbol{\alpha}$ is an invertible **L**-set, we have

$$M(\boldsymbol{\alpha}, \beta) = WR(\boldsymbol{\alpha}, \beta) = WTR(\tilde{\boldsymbol{\alpha}}, P(\tilde{\boldsymbol{\alpha}})(\beta \circ \tilde{\boldsymbol{\alpha}})).$$

This gives

$$p_{\mathbf{d}} = \sum_{\mathbf{0} \leqslant \mathbf{h} \leqslant \mathbf{d}} \langle -\mathbf{h} | (\beta \circ \tilde{\boldsymbol{\alpha}}) P(\tilde{\boldsymbol{\alpha}}) \tilde{\boldsymbol{\alpha}}^{-\mathbf{d}} \rangle b_{\mathbf{h}}$$

$$= \sum_{\mathbf{h}} \langle \mathbf{d} - \mathbf{h} | (\beta \circ \tilde{\boldsymbol{\alpha}}) P(\tilde{\boldsymbol{\alpha}}) \tilde{\boldsymbol{\alpha}}^{-\mathbf{d}} \boldsymbol{\tau}^{\mathbf{d}} > \boldsymbol{\Delta}^{\mathbf{d}-\mathbf{h}} b_{\mathbf{d}}$$

$$= (\beta \circ \tilde{\boldsymbol{\alpha}})(\boldsymbol{\Delta}) P(\tilde{\boldsymbol{\alpha}})(\boldsymbol{\Delta}) \mathbf{T}^{-\mathbf{d}} b_{\mathbf{d}}. \quad \blacksquare$$

The preceding formula can be generalized as follows:

(7.3) PROPOSITION. *Let $(\mathscr{g}_{\mathbf{d}})$ be a normalized coherent homogeneous recursive basis, with associated delta set* $\mathbf{R} := (R_s)$. *Let* $\boldsymbol{\alpha} := (\alpha_s)$ *be an invertible normalized* $\mathbf{L}$*-set and let* $(\mathscr{f}_{\mathbf{d}})$ *be a (normalized) coherent recursive basis with boundary value* β, *associated to the normalized delta set* $\mathbf{Q} := \tilde{\boldsymbol{\alpha}}(\mathbf{R})$. *Then*

$$\mathscr{f}_{\mathbf{d}} = BL\mathbf{T}^{-\mathbf{d}} \mathscr{g}_{\mathbf{d}},$$

where $\mathbf{T} = (T_s)$, *with* $R_s T_s = Q_s$,

$$B := (\beta \circ \tilde{\boldsymbol{\alpha}})(\mathbf{R}) \qquad \textit{and} \qquad L := P(\tilde{\boldsymbol{\alpha}})(\mathbf{R}).$$

Proof. The minor of coefficients of $(\mathscr{f}_{\mathbf{d}})$ with respect to the basis $(\mathscr{g}_{\mathbf{d}})$ is $M(\boldsymbol{\alpha}, \beta)$. The proof is now formally the same as that of Theorem 7.2. ∎

The next result is a generalization of the recurrence formula of the Umbral Calculus:

(7.4) PROPOSITION. *Let* $\mathbf{G} := (G_s)$, $\mathbf{Q} := (Q_s)$ *be normalized delta sets with associated homogeneous coherent recursive bases* $(\mathscr{g}_{\mathbf{d}})$ *and* $(\mathscr{q}_{\mathbf{d}})$, *respectively. Then, for every* $\mathbf{d} \in \mathbf{D}^+$, $s \in S$,

$$\mathscr{q}_{\mathbf{d}+\mathbf{e}_s} = B_s T_s^{-1} \mathscr{q}_{\mathbf{d}},$$

where T_s *is the shift-invariant operator such that* $T_s G_s = Q_s$, *and* B_s *is the linear operator such that, for every* $\mathbf{h} \in \mathbf{D}^+$,

$$B_s \mathscr{g}_{\mathbf{h}} = \mathscr{g}_{\mathbf{h}+\mathbf{e}_s}.$$

Proof. For every $\mathscr{f} \in \mathscr{F}$ such that $\langle \mathscr{f} \,|\, \mathbf{0} \rangle = 0$ we have $B_s G_s \mathscr{f} = \mathscr{f}$. Since

$$\mathscr{f}_{\mathbf{d}} = Q_s \mathscr{f}_{\mathbf{d}+\mathbf{e}_s} = G_s T_s \mathscr{f}_{\mathbf{d}+\mathbf{e}_s}$$

we get the assertion. ∎

8. THE UMBRAL CALCULUS

The classical Umbral Calculus deals with polynomial functions $f\colon \mathbb{K}^d \to \mathbb{K}$, where $\mathbb{K}$ is a field of characteristic zero and $d \in \mathbb{N}$. A sequence of polynomial functions $(f_{\mathbf{n}}(\mathbf{x}))$, where $\mathbf{n} := (n_1, n_2, ..., n_d) \in \mathbb{K}^d$ and $\mathbf{x} := (x_1, x_2, ..., x_d) \in \mathbb{K}^d$, is said to be a *sequence of binomial type* whenever $(f_{\mathbf{n}})$ is a basis for the vector space $\mathbb{K}_{[\mathbf{x}]}$ and, for every $\mathbf{x}, \mathbf{y} \in \mathbb{K}^d$ and $\mathbf{n} \in \mathbb{N}^d$,

$$f_{\mathbf{n}}(\mathbf{x} + \mathbf{y}) = \sum_{\mathbf{k} \leq \mathbf{n}} \binom{\mathbf{n}}{\mathbf{k}} f_{\mathbf{k}}(\mathbf{x}) f_{\mathbf{n}-\mathbf{k}}(\mathbf{y}),$$

where

$$\binom{\mathbf{n}}{\mathbf{k}} := \prod_{i=1}^{d} \binom{n_i}{k_i}.$$

A sequence $(g_{\mathbf{n}})$ of polynomial functions is said to be a *Sheffer sequence* related to the sequence of binomial type $(f_{\mathbf{n}})$ if $(g_{\mathbf{n}})$ is a basis for $\mathbb{K}_{[\mathbf{x}]}$ and, for every $\mathbf{x}$, $\mathbf{y} \in \mathbb{K}^d$ and $\mathbf{n} \in \mathbb{N}^d$,

$$g_n(\mathbf{x}+\mathbf{y}) = \sum_{\mathbf{k} \leqslant n} \binom{\mathbf{n}}{\mathbf{k}} g_{\mathbf{k}}(\mathbf{x}) f_{\mathbf{n}-\mathbf{k}}(\mathbf{y}).$$

Let $(f_{\mathbf{n}})$ be a sequence of binomial type and $(g_{\mathbf{n}})$ a Sheffer sequence related to $(f_{\mathbf{n}})$: it is obvious that the sequences

$$(p_{\mathbf{n}}) := (f_{\mathbf{n}}/\mathbf{n}!) \qquad \text{and} \qquad (q_{\mathbf{n}}) := (g_{\mathbf{n}}/\mathbf{n}!)$$

(here, $\mathbf{n}! := n_1!\, n_2! ... n_d!$) are bases for $\mathbb{K}_{[\mathbf{x}]}$ and satisfy the identities

$$p_{\mathbf{n}}(\mathbf{x}+\mathbf{y}) = \sum_{k \leqslant \mathbf{n}} p_{\mathbf{k}}(\mathbf{x}) p_{\mathbf{n}-\mathbf{k}}(\mathbf{y}), \tag{$*$}$$

$$q_{\mathbf{n}}(\mathbf{x}+\mathbf{y}) = \sum_{\mathbf{k} \leqslant \mathbf{n}} q_{\mathbf{k}}(\mathbf{x}) p_{\mathbf{n}-\mathbf{k}}(\mathbf{y}). \tag{$**$}$$

A sequence $(r_{\mathbf{n}})$ of polynomial functions will be called a *Pascal* (*homogeneous Pascal*) *sequence* whenever it satisfies $(**)$ ($(*)$, respectively) for every $\mathbf{x}, \mathbf{y} \in \mathbb{K}^d$ and it is a basis for $\mathbb{K}_{[\mathbf{x}]}$. Clearly, if $(r_{\mathbf{n}})$ is a Pascal sequence, then $(\mathbf{n}!\, r_{\mathbf{n}})$ is a Sheffer sequence.

We remark that, being $(p_{\mathbf{n}})$ and $(q_{\mathbf{n}})$ sequences of polynomial functions, $(**)$ holds for every $\mathbf{x}$, $\mathbf{y} \in \mathbb{K}^d$ whenever it is satisfied for every $\mathbf{x}$, $\mathbf{y} \in \mathbb{Z}^d$. Moreover, it is well known that a sequence of polynomial functions $(q_{\mathbf{n}})$, is a Pascal sequence whenever its generating function can be written as

$$\sum_{\mathbf{n}} q_{\mathbf{n}}(\mathbf{x})\, \boldsymbol{\tau}^{\mathbf{n}} = \beta \prod_i \exp(x_i \alpha_i) = \beta \exp\left(\sum_i x_i \alpha_i\right),$$

where α_i $(i = 1, 2, ..., d)$ and β are formal power series in the variables $\tau_1, \tau_2, ..., \tau_d$, such that (α_i) is an invertible $\mathbf{P}$-set and $\langle \mathbf{0} | \beta \rangle \neq 0$.

The sequence $(\mathbf{x}^n/\mathbf{n}!)$ (with $\mathbf{x}^n := x_1^{n_1} \cdots x_d^{n_d}$) is a homogeneous Pascal sequence, which is called the sequence of *divided powers*. Another notable example of a homogeneous Pascal sequence is the sequence $(\binom{\mathbf{x}}{\mathbf{n}})$, where $\mathbf{x}$ ranges over $\mathbb{K}^d$, and, for every $x \in \mathbb{K}$, $i \in \mathbb{N}$,

$$\binom{x}{i} := \frac{x(x-1) \cdots (x-i+1)}{i!}.$$

For every $f: \mathbb{K}^d \to \mathbb{K}$, let us denote by the symbol $\rho(f)$ the restriction of f to $\mathbb{Z}^d$, that is, $\rho(f) := f_{|\mathbb{Z}^d}$. It is easily realized that ρ defines an isomorphism between the vector space $\mathbb{K}_{[x]}$ of all polynomial functions in d variables over $\mathbb{K}$, and the vector space $\mathscr{F}$ of all factorial functions in d variables over $\mathbb{K}$. We explicitely note that the sequence $(\rho(\binom{x}{\mathbf{n}}))$ is the basis of Pascal functions $(\ell_{\mathbf{n}})$ of $\mathscr{F}$. Moreover, noticing that a Pascal sequence $(q_{\mathbf{n}})$ of polynomial functions in $\mathbb{K}_{[x]}$ related to the homogeneous Pascal sequence $(p_{\mathbf{n}})$ is completely defined by $(**)$ whenever the series

$$\alpha_i := \sum_{\mathbf{n}} p_{\mathbf{n}}(\mathbf{e}_i)\, \boldsymbol{\tau}^{\mathbf{n}}, \qquad i = 1, 2, \ldots, d$$

and

$$\beta := \sum_{\mathbf{n}} q_{\mathbf{n}}(\mathbf{0})\, \boldsymbol{\tau}^{\mathbf{n}}$$

are given, we can easily be convinced that ρ maps bijectively the (homogeneous) Pascal sequences into the (homogeneous) normalized recursive bases. Thus, we have a bijection between polynomial sequences of binomial and Sheffer type and normalized recursive bases of factorial functions, and the Umbral Calculus can be regarded as our theory in the case of characteristic zero.

Just as an example of this point of view, we submit two results which are typical of the case of characteristic zero.

(8.1) THEOREM. *Let $(f_{\mathbf{n}})$ be a Sheffer sequence, with generating function*

$$\sum_{\mathbf{n}} f_{\mathbf{n}}(\mathbf{x})/\mathbf{n}!\, \boldsymbol{\tau}^{\mathbf{n}} = \beta \exp\left(\sum_i x_i \alpha_i\right).$$

Then

(i) $\beta = \sum_{\mathbf{n}} f_{\mathbf{n}}(\mathbf{0})/\mathbf{n}!\, \boldsymbol{\tau}^{\mathbf{n}}$;

(ii) $\alpha_i = \log(\beta^{-1} \sum_{\mathbf{n}} f_{\mathbf{n}}(\mathbf{e}_i)/\mathbf{n}!\, \boldsymbol{\tau}^{\mathbf{n}})$, *and, hence*

(iii) $f_{\mathbf{n}}(\mathbf{x}) = \sum_{\mathbf{k}} \mathbf{n}!/\mathbf{k}!\, \langle \mathbf{n} | \boldsymbol{\alpha}^{\mathbf{k}} \beta \rangle\, \mathbf{x}^{\mathbf{k}}$.

Proof. (i) and (ii) are obvious. For (iii), we remark that the value tableaux of the recursive bases $(\rho(\mathbf{x}^{\mathbf{n}}/\mathbf{n}!))$ and $(\rho(f_{\mathbf{n}}/\mathbf{n}!))$ are $R(\exp \boldsymbol{\tau}, \upsilon)$ and $R(\exp \boldsymbol{\alpha}, \beta)$, respectively, where

$$\exp \boldsymbol{\tau} := (\exp \tau_1, \ldots, \exp \tau_d) \text{ and } \exp \boldsymbol{\alpha} := (\exp \alpha_1, \ldots, \exp \alpha_d);$$

since, by Proposition 5.1,

$$R(\exp \boldsymbol{\tau}, \upsilon) \times M(\boldsymbol{\alpha}, \beta) = R(\exp \boldsymbol{\alpha}, \beta)$$

we get the assertion. ∎

The further result we exhibit is a version of closed forms, which is a direct consequence of Propositions 7.3 and 7.4. In the sequel, we will denote by $\mathbf{D} = (D_1, \ldots, D_d)$ the set of formal partial derivatives. We have

(8.2) THEOREM. *Let $(Q_i) = \boldsymbol{\alpha}(\mathbf{D})$ be a normalized coherent delta set, and let $(f_{\mathbf{n}})$ be its associated coherent recursive basis. Let $T = (T_i)$ be the set of shift-invariant operators such that $T_i D_i = Q_i$, $i = 1, 2, \ldots, d$, and set*

$$B := \sum_{\mathbf{n}} \langle f_{\mathbf{n}} | \mathbf{0} \rangle \, \mathbf{Q}^{\mathbf{n}}$$

and

$$L := P(\boldsymbol{\alpha})(\mathbf{D}).$$

The following identities hold:

(i) $f_{\mathbf{n}}(\mathbf{x}) = BLT^{-\mathbf{n}}\mathbf{x}^{\mathbf{n}}/\mathbf{n}!$ (transfer formula),

(ii) $f_{\mathbf{n}+\mathbf{e}_i}(\mathbf{x}) = x_i/(n_i + 1)\, T_i^{-1} f_{\mathbf{n}}(\mathbf{x})$ (recurrence formula).

9. HERMITE POLYNOMIALS

The well-known Hermite polynomials $H_n(x)$ in one variable x are usually defined by means of the following generating function:

$$\sum_{n \geqslant 0} \frac{H_n(x)}{n!} t^n = \exp\left(xt - \frac{1}{2} t^2\right).$$

This polynomial sequence provides a typical example of a Sheffer sequence (see [27]).

In [13], Hermite introduced a multivariate polynomial sequence which generalizes the sequence proviously defined, as follows: given an $n \times n$ symmetric matrix $A = (a_{ij})$ with real entries, such that

(i) $\det(A) \neq 0$;

(ii) $\prod_{i=1}^{n} a_{ii} \neq 0$,

consider the quadratic form ψ associated to A, namely,

$$\psi(\mathbf{x}) = \sum_{i,j} a_{ij} x_i x_j, \qquad \mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n.$$

The sequence $(H_{\mathbf{d}}(\mathbf{x}))$ ($\mathbf{d} \in \mathbb{N}^n$, $\mathbf{x} \in \mathbb{R}^n$) of the *Hermite polynomials in n variables, of variance A* is defined by means of the generating function

$$\sum_{\mathbf{d}} \frac{H_{\mathbf{d}}(\mathbf{x})}{\mathbf{d}!} \mathbf{t}^{\mathbf{d}} = \exp\left(\frac{1}{2}\psi(\mathbf{x}) - \frac{1}{2}\psi(\mathbf{x} - \mathbf{t})\right), \qquad (*)$$

where $\mathbf{t} = (t_1, \ldots, t_n)$. Setting

$$\phi_i(\mathbf{x}) = \sum_j a_{ij} x_j, \qquad i = 1, 2, \ldots, n,$$

identity $(*)$ becomes

$$\sum_{\mathbf{d}} \frac{H_{\mathbf{d}}(\mathbf{x})}{\mathbf{d}!} \mathbf{t}^{\mathbf{d}} = \exp\left(-\frac{1}{2}\psi(\mathbf{t}) + \sum_i x_i \phi_i(\mathbf{t})\right);$$

this implies that the sequence $(H_{\mathbf{d}}(\mathbf{x})/\mathbf{d}!)$ is a Pascal sequence, with recurrence rule $(\exp(\phi_i))$ and boundary value $\exp(-\frac{1}{2}\psi)$.

Note that, if A is a diagonal matrix, the polynomials $H_{\mathbf{d}}(\mathbf{x})$ degenerate into products of n one-variable Hermite polynomials of variances $a_{11}, \ldots, a_{nn}$, respectively,

$$H_{\mathbf{d}}(\mathbf{x}) = H_{d(1)}^{[a_{11}]}(x_1) \cdots H_{d(n)}^{[a_{nn}]}(x_n),$$

and vice versa.

By the results of the preceding section, we get immediately

(9.1) PROPOSITION.

$$H_{\mathbf{d}}(\mathbf{x}) = \mathbf{d}! \sum_{\mathbf{k}} \frac{1}{\mathbf{k}!} \left\langle \mathbf{d} \,\middle|\, \boldsymbol{\phi}^{\mathbf{k}} \exp\left(-\frac{1}{2}\psi\right) \right\rangle \mathbf{x}^{\mathbf{k}}.$$

Set now

$$z_i = \phi_i(\mathbf{x}), \qquad i = 1, 2, \ldots, n,$$

that is,

$$x_i = \sum_j \frac{A_{ij}}{|A|} z_j = \tilde{\phi}_i(\mathbf{z}), \qquad i = 1, 2, \ldots, n,$$

where A_{ij} is the cofactor of a_{ij} in the matrix A, and $|A| = \det(A)$.

For every $i = 1, 2, \ldots, n$, let Z_i be the formal partial derivative with respect to z_i, namely,

$$Z_i = \tilde{\phi}_i(\mathbf{D}),$$

where $\mathbf{D} = (D_i)$, $D_i = \partial/\partial x_i$. This implies that Z_i is a shift-invariant operator for every i, and its matrix with respect to the basis of divided powers $(\mathbf{x}^{\mathbf{d}}/\mathbf{d}!)$ is the Appell matrix $M(\mathbf{t}, \tilde{\phi}_i)$.

(9.2) PROPOSITION. *For every* $\mathbf{d} \in \mathbb{N}^n$ *and for every* $i = 1, 2, \ldots, n$, *we have*

$$Z_i H_{\mathbf{d}} = \mathbf{d}(i)\, H_{\mathbf{d}-\mathbf{e}_i}.$$

Proof. It is sufficient to remark that the delta set (Z_i) is associated to the recursive basis $(H_{\mathbf{d}}/\mathbf{d}!)$. The assertion then follows from Propositions 3.10 and 9.1. ∎

Consider now the homogeneous Pascal sequence $(p_{\mathbf{d}})$ with the same recurrence rule as $(H_{\mathbf{d}}/\mathbf{d}!)$; its matrix with respect to the basis of divided powers is $M(\boldsymbol{\phi}, \upsilon)$. This implies that

$$Z_i p_{\mathbf{d}} = p_{\mathbf{d}-\mathbf{e}_i}, \qquad i = 1, 2, \ldots, n,$$

and, hence,

$$p_{\mathbf{d}} = \mathbf{z}^{\mathbf{d}}/\mathbf{d}! \qquad \text{for every } \mathbf{d}.$$

Moreover, consider the invertible shift-invariant operator

$$W = \exp(-\tfrac{1}{2}\psi(\mathbf{Z}))$$

which can be seen as a generalization of the classical Weierstrass operator; since the matrix of W with respect to the basis $(\mathbf{z}^{\mathbf{d}}/\mathbf{d}!)$ is $M(\mathbf{t}, \exp(-\frac{1}{2}\psi))$, by the results of the preceding sections, we have

$$W\mathbf{z}^{\mathbf{d}} = H_{\mathbf{d}}.$$

The following generalization of a formula of Appell and Kampé de Fériet (see [1, p. 375]) is an immediate consequence of the fact that the Pascal sequences $(H_{\mathbf{d}}/\mathbf{d}!)$ and $(\mathbf{z}^{\mathbf{d}}/\mathbf{d}!)$ have the same recurrence rule.

(9.3) PROPOSITION. *For every* $\mathbf{d} \in \mathbb{N}^n$ *and for every* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ *we have*

$$H_{\mathbf{d}}(x_1 + \tilde{\phi}_1(\mathbf{y}), \ldots, x_n + \tilde{\phi}_n(\mathbf{y})) = \sum_{\mathbf{k}} \binom{\mathbf{d}}{\mathbf{k}} \mathbf{y}^{\mathbf{k}} H_{\mathbf{d}-\mathbf{k}}(\mathbf{x}).$$

In the following, for any shift-invariant operator $T = \alpha(\mathbf{Z})$ (where α is a formal power series in n variables), set

$$T_i' = \frac{\partial \alpha}{\partial Z_i}(\mathbf{Z}), \qquad i = 1, 2, \ldots, n.$$

The following identity can be easily proved by linearity:

$$T_i' p = (T z_i - z_i T) p$$

for every polynomial p.

In particular, for the operator W previously defined, we have

$$W_i' = -\phi_i(\mathbf{Z}) W.$$

Since

$$(Wz_i - z_i W)\,\mathbf{z}^{\mathbf{d}} = W_i' \mathbf{z}^{\mathbf{d}} = -\phi_i(\mathbf{Z})\, W\mathbf{z}^{\mathbf{d}}$$

we get

$$H_{\mathbf{d}+\mathbf{e}_i} = (z_i - \phi_i(\mathbf{Z}))\, H_{\mathbf{d}}. \tag{$**$}$$

The next result generalizes the classical recurrence relations for the Hermite polynomials.

(9.4) THEOREM.

$$H_{\mathbf{d}+\mathbf{e}_i} = z_i H_{\mathbf{d}} - \sum_{j=1}^{n} a_{ij}\mathbf{d}(j)\, H_{\mathbf{d}-\mathbf{e}_j}.$$

Proof. Expand the right-hand side of $(**)$, recalling that

$$Z_i H_{\mathbf{d}} = \mathbf{d}(i)\, H_{\mathbf{d}-\mathbf{e}_i}. \quad \blacksquare$$

Our next goal is now to exhibit two generalized versions of the Rodrigues formula. To this aim, we need the following lemma, which is easily proved by induction:

(9.5) LEMMA. *Let C be a continuous derivation on the topological algebra of formal power series in n variables, with real coefficients. For every two series α, β with $C\beta \neq \zeta$, we have*

$$(C - C\beta)^n\, \alpha = \exp(\beta)\, C^n(\exp(-\beta)\, \alpha).$$

for every $n \in \mathbb{N}$.

(9.6) THEOREM. *For every* $\mathbf{d}$ *we have*

$$H_{\mathbf{d}} = \prod_{i=1}^{n} (-1)^{\mathbf{d}(i)} \exp(z_i^2/2a_{ii})\, \phi_i(\mathbf{Z})^{\mathbf{d}(i)} \exp(-z_i^2/2a_{ii}).$$

Proof. Iterating identity $(**)$, we have

$$H_{\mathbf{d}} = \prod_{i=1}^{n} (-1)^{\mathbf{d}(i)}\, (\phi_i(\mathbf{Z}) - z_i)^{\mathbf{d}(i)}\, H_0;$$

by Lemma 9.5, setting $C = \phi_i(\mathbf{Z})$ and $\beta = z_i^2/2a_{ii}$, we get the assertion. $\blacksquare$

(9.7) THEOREM. *For every* $\mathbf{d}$ *we have*

$$H_{\mathbf{d}} = (-1)^{\Sigma_i \mathbf{d}(i)} \exp(\tfrac{1}{2}\psi(\mathbf{x}))\, \mathbf{D}^{\mathbf{d}} \exp(-\tfrac{1}{2}\psi(\mathbf{x})).$$

Proof. Identity (**) can be rewritten as

$$H_{\mathbf{d}} = (-D_i + \tfrac{1}{2} D_i \psi(\mathbf{x}))\, H_{\mathbf{d}-\mathbf{e}_i};$$

iterating this last identity, we get

$$H_{\mathbf{d}} = \prod_{i=1}^{n} (-1)^{\mathbf{d}(i)} (D_i - \tfrac{1}{2}\psi(\mathbf{x}))^{\mathbf{d}(i)} H_{\mathbf{0}};$$

the assertion now follows by Lemma 9.5, setting $C = D_i$ and $\beta = \frac{1}{2}\psi(\mathbf{x})$. ∎

Finally, we give a simple proof of a multi-variate version of the formulas of Burchnall-Feldheim-Watson:

(9.8) THEOREM. *For every $i = 1, 2, \ldots, n$ and for every integer m the following identities hold*:

$$H_{\mathbf{d}+m\mathbf{e}_i} = (-1)^m \sum_{h=0}^{m} \sum_{\mathbf{j}: \Sigma \mathbf{j}(k) = h} (m)_h \binom{\mathbf{d}}{\mathbf{j}} \prod_k a_{ik}^{\mathbf{j}(k)} H_{(m-h)\mathbf{e}_i} H_{\mathbf{d}-\mathbf{j}}.$$

Proof. By (**) we have

$$H_{\mathbf{d}+m\mathbf{e}_i} = (-1)^m (\phi_i(\mathbf{Z}) - z_i)^m H_{\mathbf{d}}$$

which gives, by Lemma 9.5

$$\begin{aligned}
H_{\mathbf{d}+m\mathbf{e}_i} &= (-1)^m \exp(z_i^2/2a_{ii})\, \phi_i(\mathbf{Z})^m \exp(-z_i^2/2a_{ii})\, H_{\mathbf{d}} \\
&= (-1)^m \exp(z_i^2/2a_{ii}) \sum_{h=0}^{m} \binom{m}{h} (\phi_i(\mathbf{Z})^{m-h} \exp(-z_i^2/2a_{ii}))\, \phi_i(\mathbf{Z})^h H_{\mathbf{d}} \\
&= (-1)^m \sum_{h=0}^{m} \binom{m}{h} H_{(m-h)\mathbf{e}_i} \sum_{\mathbf{j}: \Sigma \mathbf{j}(k) = h} \binom{h}{\mathbf{j}(1) \cdots \mathbf{j}(n)} \\
&\quad \times (a_{i1} Z_1)^{\mathbf{j}(1)} \cdots (a_{in} Z_n)^{\mathbf{j}(n)} H_{\mathbf{d}} \\
&= (-1)^m \sum_{h=0}^{m} \sum_{\mathbf{j}} \binom{m}{h} \frac{h!}{\mathbf{j}(1)! \cdots \mathbf{j}(n)!} (\mathbf{d}(1))_{\mathbf{j}(1)} \cdots (\mathbf{d}(n))_{\mathbf{j}(n)} \\
&\quad \times a_{i1}^{\mathbf{j}(1)} \cdots a_{in}^{\mathbf{j}(n)} H_{(m-h)\mathbf{e}_i} H_{\mathbf{d}-\mathbf{j}}. \quad ∎
\end{aligned}$$

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